

# EFFECTIVE REPRESENTATIONS OF HECKE-KISELMAN MONOIDS OF TYPE A

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**ABSTRACT.** We prove effectiveness of certain representations of Hecke-Kiselman monoids of type  $A$  constructed by Ganyushkin and Mazorchuk and also construct further classes of effective representations for these monoids. As a consequence the effective dimension of monoids of type  $A$  is determined. We also show that odd Fibonacci numbers appear as the cardinality of certain bipartite HK-monoids and count the number of multiplicity free elements in any HK-monoid of type  $A$ .

## 1. INTRODUCTION

In [3], Kiselman noted that certain operators  $c, l, m$  in convexity theory form a monoid that can be presented as

$$G = \langle c, l, m : c^2 = c, l^2 = l, m^2 = m, \\ clc = lcl = lc, cmc = mcm = mc, lml = mlm = ml \rangle.$$

It can be noted that  $G$  is generated by three idempotents and that it has  $8 = 2^3$  idempotents.

To avoid confusion with for example representations of semigroup algebras (which do *not* appear in this thesis) we say that a semigroup representation is effective if it is injective. Note that a faithful semigroup algebra representation induces an effective semigroup representation, but that the converse is not true. The term is borrowed from [6], which deals with the (in general) very hard problem of finding a minimal effective representation for any given semigroup.

Kiselman proved that  $G$  has an effective representation by  $3 \times 3$  matrices with positive integer coefficients. This was generalised by Ganyushkin and Mazorchuk to a series  $K_n$  of monoids with  $n$  generators called Kiselmans semigroups (unpublished)

$$K_n = \langle c_1, \dots, c_n : c_i^2 = c_i \forall i, c_i c_j c_i = c_j c_i c_j = c_i c_j \forall i \leq j \rangle.$$

In this setting we have  $G = K_3$ .

Kudryavtseva and Mazorchuk later proved that  $K_n$  is generated by  $n$  idempotents and contains  $2^n$  idempotents. Moreover each  $K_n$  has an effective representation by  $n \times n$  matrices with positive integer coefficients [5]. The cardinality is in general only known to be finite [5], but we have  $|K_1|=2$ ,  $|K_2|=5$ ,  $|K_3|=18$  [3] and  $|K_4|=115$ ,  $|K_5|=1710$  [1]. Some more terms ( $|K_6| - |K_{14}|$ ) are claimed on oeis.org [8], but are not contained in any of the references.

On the other hand we can associate to every (disjoint union of) simply laced Dynkin diagram(s)  $\Gamma$  the 0–Hecke monoid  $\mathcal{H}_\Gamma$  which can be presented as

$$\mathcal{H}_\Gamma = \langle v \in V(\Gamma) : v^2 = v \forall v \in V(\Gamma), \\ v w v = w v w \forall \{v, w\} \in E(\Gamma), v w = w v \forall \{v, w\} \notin E(\Gamma) \rangle.$$

It can be shown that the semigroup algebra of  $\mathcal{H}_\Gamma$  is isomorphic to the specialization of the Hecke algebra  $\mathcal{H}_q(W_\Gamma)$  at  $q = 0$ , with  $W_\Gamma$  the Weyl group of  $\Gamma$ . This explains the name.

The notion of a Hecke-Kiselman monoid was introduced in [4] and is defined as follows

**Definition 1.** Let  $\Gamma = (V(\Gamma), E(\Gamma))$  be a simple directed graph. The *Hecke-Kiselman monoid*  $HK_\Gamma$  of  $\Gamma$  is the free semigroup  $(V(\Gamma))^*$  generated by the vertices quotiented by the relations

- (1)  $a^2 = a$  for all  $a \in V(\Gamma)$ .
- (2) If there is no edge between  $a$  and  $b$ , then  $ab = ba$ .
- (3) If  $a \rightarrow b$ , then  $aba = bab = ab$ .
- (4) If  $a \rightrightarrows b$ , then  $aba = bab$

We will call these relations (including  $a^2 = a$ ) *edge relations*.

The monoids  $HK_\Gamma$  play a central role in a paper by Grensing [2, 2012] and also appear in her Ph.D. thesis [7, 2011]. Her interest is in projection functors that turn out to obey the edge relations.

In the broader setting of Hecke-Kiselman monoids,  $K_n$  comes from the graph with  $V(\Gamma) = \{1, 2, \dots, n\}$  and  $E(\Gamma) = \{(i, j) | i < j\}$ . As  $HK_\Gamma$  can be viewed as a generalisation of  $K_n$ , it is reasonable to ask the following questions

- (1) Does  $HK_\Gamma$  have  $2^n$  idempotents, where  $n$  is the number of vertices? If no, then how many?
- (2) Does  $HK_\Gamma$  have an effective representation by  $n \times n$  matrices? With positive integer coefficients?
- (3) Is  $HK_\Gamma$  finite? If so, can we find the number of elements?

Every graph in this thesis is assumed to have no double edges  $a \rightrightarrows b$ , unless explicitly stated otherwise. This is mainly because the type of representation we will study is useless for graphs with double edges  $a \rightrightarrows b$  (see Remark 6). We will use the notation  $\underline{n} = \{1, 2, \dots, n\}$ .

*Remark 2.* It is easy to see that if  $\Gamma = \Gamma_1 \sqcup \Gamma_2$  we have that every vertex from  $\Gamma_1$  commutes with every vertex from  $\Gamma_2$  and hence  $HK_\Gamma = HK_{\Gamma_1} \oplus HK_{\Gamma_2}$  as monoids. Hence we only need to study connected graphs.

Since elements in  $HK_\Gamma$  are equivalence classes of words in the alphabet  $V(\Gamma)$  of vertices, we will denote by  $[\mathbf{w}] \in HK_\Gamma$  an element and  $\mathbf{w}' \in [\mathbf{w}]$  a word, i.e. a representative of that equivalence class. We will use bold fonts to distinguish general words from vertices, i.e.  $a \in V(\Gamma)$  is a vertex, but  $\mathbf{w} \in (V(\Gamma))^*$  is a word. The empty word will be denoted  $\varepsilon$ . We will make use of two binary relations on  $(V(\Gamma))^*$ . Let  $\mathbf{w} \approx \mathbf{w}'$  be defined as “ $\mathbf{w}$  and  $\mathbf{w}'$  differ by at most one edge relation”, and  $\mathbf{w} \sim \mathbf{w}' : \iff [\mathbf{w}] = [\mathbf{w}']$ . Note that the relation  $\sim$  is the transitive closure of the relation  $\approx$  and thus in particular  $\mathbf{w} \approx \mathbf{w}'$  is a stronger statement than  $\mathbf{w} \sim \mathbf{w}'$ .

It is easy to see that all edge relations preserve existence of a given vertex in a word (while multiplicity will vary). Thus existence of a particular vertex is also a property of elements in  $HK_\Gamma$ .

**Definition 3.** The set of vertices that are present in an element  $[\mathbf{w}] \in HK_\Gamma$  is denoted by  $\mathfrak{c}[\mathbf{w}]$  and is called the content of  $[\mathbf{w}]$ . The set of elements  $[\mathbf{w}]$  which have  $\mathfrak{c}[\mathbf{w}] = V(\Gamma)$  will be denoted by  $Max_\Gamma$  and it's cardinality by  $\mathfrak{m}(\Gamma)$ .

Let  $\mathcal{R}$  be a unital commutative ring without zero divisors and  $W = \bigoplus_{v \in V(\Gamma)} \mathcal{R}v$ . Let  $v_i, i \in \underline{n}$ , be an enumeration of the vertices in  $\Gamma$  and let

$$\theta_i^f(v_j) = \begin{cases} v_j, & i \neq j \\ \sum_{v_k \rightarrow v_i} f(v_k \rightarrow v_i) v_k, & i = j \end{cases}$$

for a fixed function  $f : E(\Gamma) \rightarrow \mathcal{R}$ . For readability, we assume the notation  $f_{ki} := f(v_k \rightarrow v_i)$  and  $\theta_i = \theta_i^f$ , when  $f$  is clear from context. We define the function  $R_f : V(\Gamma) \rightarrow \text{End}_{\mathcal{R}}(W)$  by  $v_i \mapsto \theta_i^f$  and extend it to words in the obvious way, namely that juxtaposition becomes composition  $R_f(\mathbf{vw}) = R_f(\mathbf{v}) \circ R_f(\mathbf{w})$ . The endomorphisms  $\theta_i$  will be referred to as *atomic*.

The following theorem is a straightforward generalisation of “linear integral representations” in [4, 3.2] that nevertheless needs to be proved.

**Theorem 4.**  $R_f$  is well-defined as a map from  $HK_\Gamma \rightarrow \text{End}_{\mathcal{R}}(W)$  and is a representation.

*Proof.* Since  $R_f$  is defined as a composition of atomic endomorphisms, it is a representation if it is welldefined as a map  $HK_\Gamma \rightarrow \text{End}_{\mathcal{R}}(W)$ . We need to show that  $R_f$  respects the edge relations.

$$\text{Relation } a^2 = a. \theta_i \circ \theta_i(v_j) = \begin{cases} v_j, & j \neq i \\ \sum_{v_k \rightarrow v_i} f_{ki} \theta_i(v_k), & j = i \end{cases}$$

But since  $\Gamma$  is loop free,  $v_k \rightarrow v_i \Rightarrow k \neq i$  and thus  $\theta_i(v_k) = v_k$ . Thus  $\theta_i^2 = \theta_i$  for all  $i \in \underline{n}$ .

Relation  $ab = ba$  for  $a$  and  $b$  non-adjacent.

$$\theta_i \circ \theta_j(v_k) = \begin{cases} \theta_i(v_k), & k \neq j \\ \sum_{v_l \rightarrow v_j} f_{lj} \theta_i(v_l), & k = j \end{cases} = \begin{cases} v_k, & k \neq j, i \\ \sum_{v_l \rightarrow v_i} f_{li} v_l, & k = i \\ \sum_{v_l \rightarrow v_j} f_{lj} v_l, & k = j \end{cases}$$

The reason that  $\theta_i(v_l) = v_l$  is that  $v_l \rightarrow v_j$  and  $v_i \not\rightarrow v_j$ . Since the right hand side is symmetrical we get  $\theta_i \circ \theta_j = \theta_j \circ \theta_i$  for all  $i, j$  such that  $v_i$  and  $v_j$  non-adjacent.

Relation  $aba = bab = ab$  for  $a \rightarrow b$ .

$$\theta_i \circ \theta_j \circ \theta_i(v_k) = \begin{cases} \theta_i \circ \theta_j(v_k), & k \neq i \\ \sum_{v_l \rightarrow v_i} f_{li} \theta_i \circ \theta_j(v_l), & k = i \end{cases} = \begin{cases} v_k, & k \neq i, j \\ \sum_{v_l \rightarrow v_j} f_{lj} \theta_i(v_l), & k = j \\ \sum_{v_l \rightarrow v_i} f_{li} v_l, & k = i \end{cases}$$

Since  $v_i \rightarrow v_j$  we have

$$\sum_{v_l \rightarrow v_j} f_{lj} \theta_i(v_l) = \sum_{v_l \rightarrow v_j, l \neq i} f_{lj} v_l + f_{ij} \sum_{v_l \rightarrow v_i} f_{li} v_l$$

On the other hand, we have

$$\theta_j \circ \theta_i \circ \theta_j(v_k) = \begin{cases} \theta_j \circ \theta_i(v_k), & k \neq j \\ \sum_{v_l \rightarrow v_j} f_{lj} \theta_j \circ \theta_i(v_l), & k = j \end{cases} = \begin{cases} v_k, & k \neq i, j \\ \sum_{v_l \rightarrow v_i} f_{li} v_l, & k = i \\ \sum_{v_l \rightarrow v_j} f_{lj} \theta_j \circ \theta_i(v_l), & k = j \end{cases}$$

But

$$\sum_{v_l \rightarrow v_j} f_{lj} \theta_j \circ \theta_i(v_l) = \sum_{v_l \rightarrow v_j, l \neq i} f_{lj} \theta_j(v_l) + f_{ij} \sum_{v_l \rightarrow v_i} f_{li} \theta_j(v_l).$$

Since  $(v_l \rightarrow v_j \text{ or } v_l \rightarrow v_i) \Rightarrow v_l \neq v_j$  we have

$$\sum_{v_l \rightarrow v_j} f_{lj} \theta_j \circ \theta_i(v_l) = \sum_{v_l \rightarrow v_j, l \neq i} f_{lj} v_l + f_{ij} \sum_{v_l \rightarrow v_i} f_{li} v_l.$$

$$\theta_i \circ \theta_j(v_k) = \begin{cases} \theta_i(v_k), & k \neq j \\ \sum_{v_l \rightarrow v_j} f_{lj} \theta_i(v_l), & k = j \end{cases} = \begin{cases} v_k, & k \neq i, j \\ \sum_{v_l \rightarrow v_i} f_{li} v_l, & k = i \\ \sum_{v_l \rightarrow v_j} f_{lj} \theta_i(v_l), & k = j \end{cases}$$

But

$$\sum_{v_l \rightarrow v_j} f_{lj} \theta_i(v_l) = \sum_{v_l \rightarrow v_j, l \neq i} f_{lj} v_l + f_{ij} \sum_{v_l \rightarrow v_i} f_{li} v_l.$$

Thus  $\theta_i \circ \theta_j \circ \theta_i = \theta_j \circ \theta_i \circ \theta_j = \theta_i \circ \theta_j$  for  $v_i \rightarrow v_j$ .  $\square$

In the case  $\Gamma = 1 \rightarrow 2 \rightarrow \dots \rightarrow n$ ,  $f_{i(i+1)} \equiv 1$  and  $\mathcal{R} = \mathbb{Z}$  we have that  $R_f$  is effective, as proved in [4]. We will by  $R_k$  denote the unique (for each graph and each ring) representation  $R_f$  with  $f(v_i \rightarrow v_j) \equiv k$ .

**Lemma 5.** *Let  $\Gamma = 1 \rightarrow 2 \rightarrow \dots \rightarrow n$ . Then any choice of  $f : E(\Gamma) \rightarrow \mathcal{R}$  such that  $f_{i(i+1)} \neq 0$  for all  $i \in \underline{n-1}$  gives an effective representation  $R_f$  of  $HK_\Gamma$ .*

*Proof.* First observe that  $R_f[\mathbf{w}](x) = cx'$  is a sum of at most one non-zero terms. Let  $x' = x_i \rightarrow x_{i+1} \rightarrow \dots \rightarrow x_j = x$  be the unique oriented path from  $x'$  to  $x$ . Then  $R_f[\mathbf{w}](x) = cx' \Rightarrow R_f[\mathbf{w}](x) = \theta_i \circ \theta_{i+1} \circ \dots \circ \theta_j(x)$ . Thus the only thing that differ between  $R_f[\mathbf{w}]$  and  $\theta_i \circ \theta_{i+1} \circ \dots \circ \theta_j$  is endomorphisms that act like unity on  $x$ .

Since this is done using only  $f_{ij} \neq 0$  the map  $cx \mapsto x, c \neq 0$  is injective. Indeed, for  $R_1[\mathbf{w}](x) = x'$  we have  $R_f[\mathbf{w}](x) = (\prod_{k=i}^{j-1} f_{k(k+1)})x'$  which given  $c_{x,x'} = \prod_{k=i}^{j-1} f_{k(k+1)}$  constructed from the unique path gives the inverse  $x \mapsto c_{x,x'}x'$  which proves the claim. Recall that  $\mathcal{R}$  is assumed to have no zero divisors.  $\square$

*Remark 6.* If  $\Gamma$  contains a double edge we have that  $R_f$  in general is not a representation, and if it is, it is not effective. Indeed, if the graph  $v_1 \rightleftharpoons v_2$  is a subgraph of  $\Gamma$  we should have that  $\theta_1 \circ \theta_2 \circ \theta_1 = \theta_2 \circ \theta_1 \circ \theta_2$ . Let us simplify notation by setting  $S_i = \sum_{v_k \rightarrow v_i, k \neq 1,2} f_{ki} v_k$  for  $i \in \{1,2\}$ . Note that  $\theta_i(S_j) = S_j$  for  $i, j \in \{1,2\}$ .

$\theta_1 \circ \theta_2 \circ \theta_1(v_1) = \theta_1 \circ \theta_2(f_{21}v_2 + S_1) = \theta_1(f_{12}f_{21}v_1 + S_1 + S_2) = f_{21}^2 f_{12}v_2 + 2S_1 + S_2$  and

$$\theta_2 \circ \theta_1 \circ \theta_2(v_1) = \theta_2 \circ \theta_1(v_1) = \theta_2(c_{21}v_2 + S_1) = f_{12}f_{21}v_1 + S_1 + S_2$$

Since the left hand sides coincide, so do the right hand sides.

$$f_{21}^2 f_{12}v_2 + 2S_1 + S_2 = f_{12}f_{21}v_1 + S_1 + S_2 \iff f_{21}^2 f_{12}v_2 + S_1 = f_{12}f_{21}v_1$$

This means that  $f_{12}f_{21} = 0 = S_1$ , which since  $R$  has no zero divisors implies that  $f_{12} = 0$  or  $f_{21} = 0$ . By symmetry we get  $S_2 = 0$ . W.l.o.g. assume  $f_{12} = 0$ . Then

$$\theta_2 \circ \theta_1(v_1) = \theta_2(f_{21}v_2 + S_1) = f_{12}f_{21}v_1 = 0 = \theta_1 \circ \theta_2 \circ \theta_1(v_1)$$

and

$$\theta_2 \circ \theta_1(v_2) = \theta_2(v_2) = f_{12}v_1 + S_2 = 0 = \theta_1 \circ \theta_2 \circ \theta_1(v_2)$$

Since  $\theta_i(v_k) = v_k$  for  $i \in \{1, 2\}$  and  $k \neq 1, 2$  we have that

$$R_f[v_1 v_2 v_1] = \theta_1 \circ \theta_2 \circ \theta_1 = \theta_2 \circ \theta_1 = R_f[v_2 v_1].$$

But since  $v_1 v_2 v_1 \neq v_2 v_1$  ( $[v_2 v_1] = \{v_2^{k+1} v_1^{l+1} | k, l \in \mathbb{N}\}$  does not include  $v_1 v_2 v_1$ ) we have that  $R_f$  is not effective.

**Theorem 7.** *Let  $C \subset \Gamma$  be an oriented cycle of length  $l \geq 3$  and let  $[\mathbf{w}] \in HK_\Gamma$  be an element such that  $V(C) \subset \mathbf{c}[\mathbf{w}]$ . Then  $[\mathbf{w}^k]$  are different elements for all  $k \in \mathbb{N}$ .*

*Proof.* Let  $p : HK_\Gamma \rightarrow HK_C$  be the projection that removes all vertices and edges not in  $C$ . If we prove that  $p[\mathbf{w}^k] \in HK_C$  are different for all  $k \in \mathbb{N}$ , then so are  $[\mathbf{w}^k] \in HK_\Gamma$ , so it is sufficient to prove the claim for  $\Gamma = C = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_l \rightarrow v_1$ . For simple notation, we adopt cyclic indexing, i.e the indices  $i \in \mathbb{Z}_l$  add modulo  $l$ .

We will need a representation to separate different elements, and we choose  $R_2$  with the ground ring  $\mathcal{R} = \mathbb{Z}$ . Let  $[\mathbf{w}] \in Max_C$ . Since each atomic representation  $\theta_i$  takes  $v_i$  to  $2v_{i-1}$  and  $[\mathbf{w}]$  contains every vertex we get that  $R_2[\mathbf{w}](v_i) = 2^{m_i} v_{t(i)}$ , where  $m_i \geq 1$  and  $t$  is a transformation on the set  $\underline{l}$ . By forgetting momentarily the transformation in the indices and concentrating on the coefficient we see that  $R_2[\mathbf{w}^k](v_i) = 2^{m_{i_1}} 2^{m_{i_2}} \dots 2^{m_{i_k}} v_{T(i)}$ , i.e. that each application of  $R_2[\mathbf{w}]$  raises the coefficient. In particular we have that  $R_2[\mathbf{w}^{k_1}](v_i) \neq R_2[\mathbf{w}^{k_2}](v_i)$  for  $k_1 \neq k_2$ , which implies  $R_2[\mathbf{w}^{k_1}] \neq R_2[\mathbf{w}^{k_2}]$  for  $k_1 \neq k_2$ .  $\square$

**Corollary 8.** *The number of idempotents in  $HK_\Gamma$  equals the number of (directed, length  $l \geq 3$ ) cycle free full subgraphs of  $\Gamma$ .*

*Proof.* For a subgraph  $\Gamma'$  not containing any directed cycle of length  $l \geq 3$  we have that  $HK_{\Gamma'}$  is a quotient of the Kiselman monoid by certain edge relations. Indeed, if we view the graph as a relation on the edges and take the transitive closure, we get a partial order  $\prec$ . Since any partial order can be extended to a total order, this induces an inclusion  $\Gamma' \subset G_m$ , where  $G_m$  is the graph of the Kiselman semigroup  $K_m$  (the graph of a Hecke-Kiselman monoid is unique up to isomorphism by [4]).

Since edge relations respect content and  $K_m$  has precisely one idempotent for each content [5] it follows that  $HK_{\Gamma'}$  has precisely one idempotent of maximal content. Thus  $HK_\Gamma$  has precisely one idempotent with content  $V(\Gamma')$ .

For a subgraph  $\Gamma'$  containing a directed cycle of length  $l \geq 3$  we are in the situation described by the previous theorem and thus  $HK_{\Gamma'}$  has no idempotent of maximal content. Equivalently,  $HK_\Gamma$  has no idempotent with content  $V(\Gamma')$ .  $\square$

**Definition 9.** Let  $\Gamma$  be obtained from the Dynkin diagram  $A_n$  by choosing a direction on every edge. Then  $\Gamma$  is said to be of *type  $A_n$* . Because of the bijectivity (up to isomorphism) between graphs and their Hecke-Kiselman monoids we can define a Hecke-Kiselman monoid to be of type  $A_n$  if its corresponding graph is of type  $A_n$ .

The same definition is used in [4], but with the notation “ $\Gamma$  is of type  $A$ ” instead of “ $\Gamma$  is of type  $A_n$ ”.

$$a_1 \longrightarrow a_2 \longleftarrow a_3 \longleftarrow \dots \longrightarrow a_n$$

FIGURE 1.1. An example of a graph of type  $A_n$

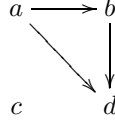


FIGURE 1.2. In this graph  $\deg_+(a) = 2, \deg_+(b) = 1, \deg_+(c) = \deg_+(d) = 0$

**Definition 10.** For any directed graph  $\Gamma$  we have the indegree function  $\deg_-(v) : V(\Gamma) \rightarrow \mathbb{N}$ , which counts how many edges that point towards  $v$ , and the outdegree function  $\deg_+(v) : V(\Gamma) \rightarrow \mathbb{N}$ , which counts how many edges that point from  $v$ .

**Definition 11.** A vertex  $v$  satisfying  $\deg_+(v)\deg_-(v) = 0$  is called a *kink*. Equivalently  $v$  is a kink iff  $v$  is a source or  $v$  is a sink.

**Lemma 12.** Any element of  $[\mathbf{w}'] \in HK_\Gamma$  contains a word  $\mathbf{w}$  where all kinks appear with multiplicity at most 1. This will later be phrased as “ $\mathbf{w}$  is multiplicity free with respect to  $a$ ” or “ $a$  is multiplicity free in  $\mathbf{w}$ ”.

*Proof.* Let  $\overleftarrow{\Gamma}$  be the graph obtained from  $\Gamma$  by reversing all arrows. By [4] we have that  $HK_\Gamma$  is anti-isomorphic with  $HK_{\overleftarrow{\Gamma}}$ . Thus by reversing every word  $\mathbf{w}$  we may assume any given kink to be a source. Let  $\mathbf{w}$  be a word with more than one occurrence of a kink  $a$ . For any  $x \in V(\Gamma)$  we have one of the following

- (1)  $x = a \Rightarrow ax = aa = aaa = axa$
- (2)  $a \rightarrow x \Rightarrow ax = axa$
- (3)  $a$  and  $x$  are nonadjacent  $\Rightarrow ax = aax = axa$

Thus  $ax = axa$  for all  $x \in V(\Gamma)$

$$\begin{aligned} \mathbf{w} &= \mathbf{w}_1 a x_1 x_2 \cdots x_n a \mathbf{w}_2 \approx \mathbf{w}_1 a x_1 a x_2 \cdots x_n a \mathbf{w}_2 \approx \mathbf{w}_1 a x_1 a x_2 a \cdots x_n a \mathbf{w}_2 \approx \cdots \\ &\approx \mathbf{w}_1 a x_1 a x_2 a \cdots a x_n a a \mathbf{w}_2 \approx \mathbf{w}_1 a x_1 a x_2 a \cdots a x_n a \mathbf{w}_2 \approx \cdots \approx \mathbf{w}_1 a x_1 x_2 \cdots x_n \mathbf{w}_2 \end{aligned}$$

This gives us an equivalent word which up to deletion of one  $a$  is precisely  $\mathbf{w}$ . As long as there are multiple copies we repeat the procedure. What remains in the end is  $\mathbf{w}$  with all but the leftmost occurrence of  $a$  removed (for sources) or all but the rightmost occurrence of  $a$  (for sinks). Since no other letter was changed, we can delete kinks independantly of each other.  $\square$

**Definition 13.** Let  $A \subset V(\Gamma)$  be a set of vertices. We define  $\mathcal{MF}_A \subset (V(\Gamma))^*$  as the set of words which are multiplicity free with respect to all vertices in  $A$ .

**Lemma 14.** Let  $A$  be a set of kinks (not necessarily all) in  $\Gamma$ . Given two words  $\mathbf{w} \sim \mathbf{w}'$  such that  $\mathbf{w}, \mathbf{w}' \in \mathcal{MF}_A$  we can find a series of words  $\mathbf{w} = \mathbf{w}_1 \approx \cdots \approx \mathbf{w}_k = \mathbf{w}'$  where  $\mathbf{w}_i \in \mathcal{MF}_A$  for all  $i$ .

*Proof.* By the definition of the relation  $\sim$  there exists a series of words

$$\mathbf{w} = \mathbf{w}_1 \approx \mathbf{w}_2 \approx \cdots \approx \mathbf{w}_k = \mathbf{w}',$$

so what remains to be shown is that  $\mathbf{w}_i$  can be assumed to be in  $\mathcal{MF}_A$ .

Let us first concentrate on one kink  $a \in A$ . As can be seen in the previous lemma we may remove all but the leftmost or all but the rightmost appearances

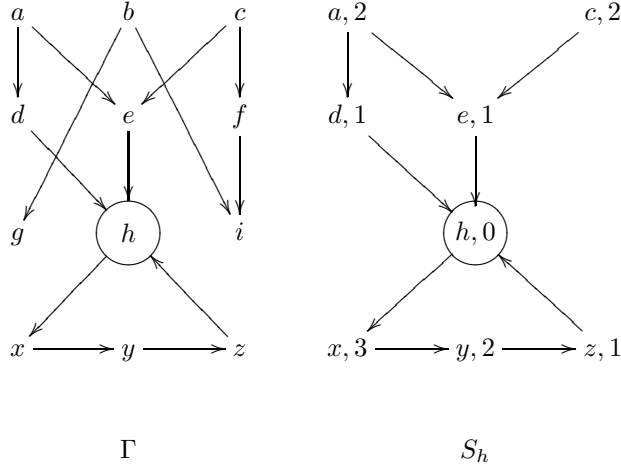


FIGURE 1.3. The source graph  $S_h$  of a vertex  $h \in \Gamma$ . The numbers at each vertex in  $S_h$  indicate the first  $V_i$  which the vertex appeared in. In this case, both a directed cycle  $h \leftarrow z \leftarrow y \leftarrow x \leftarrow h$  and an oriented cycle  $a \rightarrow e \rightarrow h \leftarrow d \leftarrow a$  appear.

of  $a$  depending on if  $a$  is a sink or a source. Let  $\psi_a$  be a function on words which removes all superfluous appearances of  $a$  accordingly. W.l.o.g. assume we keep the leftmost  $a$ . Note that  $\psi_a(\mathbf{w}) = \mathbf{w}$  and  $\psi_a(\mathbf{w}') = \mathbf{w}'$ . We need to split into cases depending on the relation used between  $\mathbf{w}_i$  and  $\mathbf{w}_{i+1}$

- (1)  $x^2 = x, xy = yx$  or  $xyx = yxy = xy$  for  $x, y \neq a$  will give  $\mathbf{w}_i \approx \mathbf{w}_{i+1} \Rightarrow \psi_a(\mathbf{w}_i) \approx \psi_a(\mathbf{w}_{i+1})$  with the same relation as the difference.
- (2)  $a^2 = a, ax = xa$  or  $axa = xax = ax$  when the leftmost  $a$  is *not* involved gives  $\varepsilon = \varepsilon, x = x$  or  $x = x^2 = x$  under  $\psi_a$  and thus  $\mathbf{w}_i \approx \mathbf{w}_{i+1} \Rightarrow \psi_a(\mathbf{w}_i) \approx \psi_a(\mathbf{w}_{i+1})$ .
- (3)  $a^2 = a, ax = xa$  or  $axa = xax = ax$  when the leftmost  $a$  is involved gives  $a = a, ax = xa$  or  $ax = xax = ax$  under  $\psi_a$  and thus  $\mathbf{w}_i \approx \mathbf{w}_{i+1} \Rightarrow \psi_a(\mathbf{w}_i) \approx \psi_a(\mathbf{w}_{i+1})$ .

To get a series which is multiplicity free for all chosen kinks  $a_i, i \in I$  we form  $\psi_{a_i}$  in the same manner and set  $\psi = \psi_{a_1} \circ \psi_{a_2} \circ \dots \circ \psi_{a_j}$ . Then

$$\mathbf{w} = \psi(\mathbf{w}) = \psi(\mathbf{w}_0) \approx \psi(\mathbf{w}_1) \approx \dots \approx \psi(\mathbf{w}_k) = \psi(\mathbf{w}') = \mathbf{w}'$$

is a series which satisfies our claim. □

**Definition 15.** Given a graph  $\Gamma$  and a vertex  $a$  we define the *source graph*  $S_a$  as the full subgraph of  $\Gamma$  with vertices  $V(S_a) = \{v \in \Gamma \mid \text{there is a directed path from } v \text{ to } a\}$ . Alternatively we can define the vertex set  $V(S_a)$  recursively as

- (1)  $V_0 = \{a\}$
- (2)  $V_{i+1} = V_i \cup \{v \in \Gamma \mid \exists u \in V_i : v \rightarrow u\}$

For some  $N \leq n = |\Gamma|$  we have that  $V_{N+k} = \dots = V_{N+1} = V_N$  for all  $k \in \mathbb{N}$  and we set  $V(S_a) = V_N$ . Note that the source graph can contain cycles.

**Lemma 16.** For  $a \in \Gamma$  let  $p_a : HK_\Gamma \rightarrow HK_{S_a}$  be the projection which removes vertices not in the source graph of  $a$ . Then for  $[\mathbf{w}] \in HK_\Gamma$  and a function  $f : E(\Gamma) \rightarrow \mathcal{R}$  we have that  $R_f[\mathbf{w}](a) = R_f p_a[\mathbf{w}](a)$ .

*Proof.* By the construction of  $R_f$  we have  $R_f[\mathbf{w}](a) = \sum_{v \in S_a} c_v v$  for some constants  $c_v \in \mathcal{R}$  for any element  $[\mathbf{w}] \in HK_\Gamma$ . Let  $\mathbf{w} = \mathbf{w}_1 x \mathbf{w}_2$  be a word with  $x \notin S_a$ . Then

$$\begin{aligned} R_f[\mathbf{w}_1 x \mathbf{w}_2](a) &= R_f[\mathbf{w}_1] R_f[x] R_f[\mathbf{w}_2](a) = \\ &= R_f[\mathbf{w}_1] R_f[x] \sum_{v \in S_a} c_v v = R_f[\mathbf{w}_1] \sum_{v \in S_a} c_v R_f[x](v). \end{aligned}$$

But since  $x \notin S_a$  and  $v \in S_a$  we have  $x \neq v$  and thus  $R_f[x](v) = v$  by definition of  $R_f$ . Thus

$$R_f[\mathbf{w}_1] \sum_{v \in S_a} c_v R_f[x](v) = R_f[\mathbf{w}_1] \sum_{v \in S_a} c_v v = R_f[\mathbf{w}_1] R_f[\mathbf{w}_2](a) = R_f[\mathbf{w}_1 \mathbf{w}_2](a).$$

As long as we have  $x \notin S_a$  present in  $\mathbf{w}$  we can cancel it with the same resulting action on  $a$ . Eventually we will get to  $p_a[\mathbf{w}]$  and have  $R_f[\mathbf{w}](a) = R_f p_a[\mathbf{w}](a)$ .  $\square$

**Definition 17.** For  $\Gamma' \subset \Gamma$  a full subgraph and a representation  $R_f : HK_\Gamma \rightarrow \text{End}_{\mathcal{R}}(W)$  there is one canonical representation  $R'_f : HK_{\Gamma'} \rightarrow \text{End}_{\mathcal{R}}(\bigoplus_{v \in V(\Gamma')} \mathcal{R}v)$  defined on the atomic representations as

$$\theta'_i(v_j) = \begin{cases} v_j, & i \neq j \\ \sum_{v_k \rightarrow v_i, v_k \in V(\Gamma')} f_{ki} v_k, & i = j \end{cases}$$

We say that  $R'_f$  is *restricted* from  $R_f$ .

If  $\Gamma = \bigcup_{i \in I} \Gamma_i$  is an edge disjoint union and  $R_{f_i} : HK_{\Gamma_i} \rightarrow \text{End}_{\mathcal{R}}(\bigoplus_{v \in V(\Gamma_i)} \mathcal{R}v)$  are representation there is one unique representation  $R_f : HK_\Gamma \rightarrow \text{End}_{\mathcal{R}}(W)$  defined by  $f(v_j \rightarrow v_k) = f_i(v_j \rightarrow v_k)$  for the unique  $i$  such that the edge  $v_j \rightarrow v_k$  lies in  $\Gamma_i$ . We say that  $R_f$  is *up induced* from  $\{R_{f_i}\}_{i \in I}$ .

**Theorem 18.** For  $\Gamma' \subset \Gamma$  a full subgraph, let  $p_{V(\Gamma')} : W \rightarrow \bigoplus_{v \in \Gamma'} \mathcal{R}v$  be the canonical projection in modules and  $p : HK_\Gamma \rightarrow HK_{\Gamma'}$  the canonical projection in monoids. Assume that every directed path that starts and ends in  $\Gamma'$  is completely contained in  $\Gamma'$ . Then  $p_{V(\Gamma')} \circ R_f[\mathbf{w}]|_{V(\Gamma')} = R'_f p[\mathbf{w}]$ , where  $R'_f$  is restricted from  $R_f$  (as in the definition above).

*Proof.* We prove the claim by showing that it holds for atomic representations and compositions with one of the representations beeing atomic.

$$\begin{aligned} p_{V(\Gamma')} \theta_i(v_j) &= \begin{cases} p_{V(\Gamma')}(v_j), & i \neq j \\ p_{V(\Gamma')}(\sum_{v_k \rightarrow v_i} f_{ki} v_k), & i = j \end{cases} \\ &= \begin{cases} v_j, & i \neq j, v_j \in \Gamma' \\ 0, & i \neq j, v_j \notin \Gamma' \\ \sum_{v_k \rightarrow v_i, v_k \in V(\Gamma')} f_{ki} v_k, & i = j \end{cases} \end{aligned}$$

But we also have

$$\begin{aligned} &= R'_f p[v_i](v_j) = \begin{cases} R'_f[v_i](v_j), & v_i \in \Gamma' \\ 0, & v_i \notin \Gamma' \end{cases} \\ &= \begin{cases} v_j, & i \neq j, v_i \in \Gamma' \\ \sum_{v_k \rightarrow v_i, v_k \in V(\Gamma')} f_{ki} v_k, & i = j, v_i \in \Gamma' \\ 0, & i = j, v_i \notin \Gamma' \end{cases} \end{aligned}$$



Fix some  $a \in \Gamma'$  and let  $S_a^{in} := S_a \cap \Gamma'$  and  $S_a^{out} := S_a \setminus \Gamma'$ . Every edge between  $v_{out} \in S_a^{out}$  and  $v_{in} \in S_a^{in}$  has to be in the direction  $v_{out} \rightarrow v_{in}$ . Indeed, if there is an edge  $v_{in} \rightarrow v_{out}$  we can add it to the path  $v_{out} \rightarrow v_1 \rightarrow \dots \rightarrow v_k \rightarrow a$  (part of the definition of  $S_a$ ) to obtain  $v_{in} \rightarrow v_{out} \rightarrow v_1 \rightarrow \dots \rightarrow v_k \rightarrow a$ , a directed path which starts and ends in  $\Gamma'$ , but has at least one vertex  $v_{out}$  not in  $\Gamma'$ , which would contradict our assumption. Thus  $\theta_i(v_{out}) = \sum_{v_k \rightarrow v_{out}, v_k \notin \Gamma} c_{v_k} v_k$  for all  $v_{out} \in S_a^{out}$ . Now assume  $p_{V(\Gamma')} \circ R_f[\mathbf{w}]|_{V(\Gamma')} = R'_f p[\mathbf{w}]$  for some  $[\mathbf{w}] \in HK_\Gamma$ . Then

$$\begin{aligned} p_{V(\Gamma')} \circ R_f[v_j \mathbf{w}]|_{V(\Gamma')}(v_i) &= p_{V(\Gamma')} \left( \sum_{v_k \rightarrow v_i} f_{ki} \theta_j(v_k) \right) = \\ &= \sum_{v_k \rightarrow v_i} f_{ki} p_{V(\Gamma')}(\theta_j(v_k)) = \sum_{v_k \rightarrow v_i, v_k \in \Gamma} f_{ki} p_{V(\Gamma')}(\theta_j(v_k)). \end{aligned}$$

On the other hand,

$$R'_f p[v_j \mathbf{w}](v_i) = \sum_{v_k \rightarrow v_i, v_k \in V(\Gamma')} f_{ki} \theta'_j(v_k),$$

and  $p_{V(\Gamma')} \circ \theta_j = \theta'_j$ , so we have the desired equality.  $\square$

## 2. SERIAL DECOMPOSITION

We may try to understand a large graph and it's corresponding semigroup in terms of subgraphs.

**Definition 19.** Let  $\Gamma = \bigcup_{i \in \underline{n}} \Gamma_i$  be a graph which can be described as an edge disjoint union of graphs that each share one vertex with the next (a cut vertex) and such that each such vertex is a kink. We call such a graph a *chain* and the subgraphs  $\Gamma_i$  *links*. We enumerate the cut vertices as  $a_i \in \Gamma_i \cap \Gamma_{i+1}$ . By  $R_{f_i}$  we mean a representation corresponding to the link  $\Gamma_i$ . By  $\Gamma'_i = \Gamma_i \setminus \{a_{i-1}, a_i\}$  we mean the interior of a link.

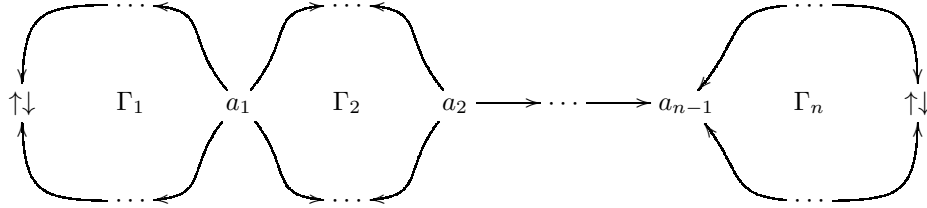


FIGURE 2.1. An example of a chain. The signs  $s(a_i)$  may come in any order.

For each link we have the projection  $p_i : HK_\Gamma \rightarrow HK_{\Gamma_i}$  which is obtained by introducing the relations  $x = \varepsilon$  for all  $x \in V(\Gamma \setminus \Gamma_i)$ . We combine them to the projection  $p : HK_\Gamma \rightarrow HK_{\Gamma_1} \times HK_{\Gamma_2} \times \dots \times HK_{\Gamma_n}$ ,  $p[\mathbf{w}] = (p_1[\mathbf{w}], p_2[\mathbf{w}], \dots, p_n[\mathbf{w}])$ .

**Theorem 20.** *Let  $\Gamma$  be a chain and  $p$  as above. Then  $p$  is injective.*

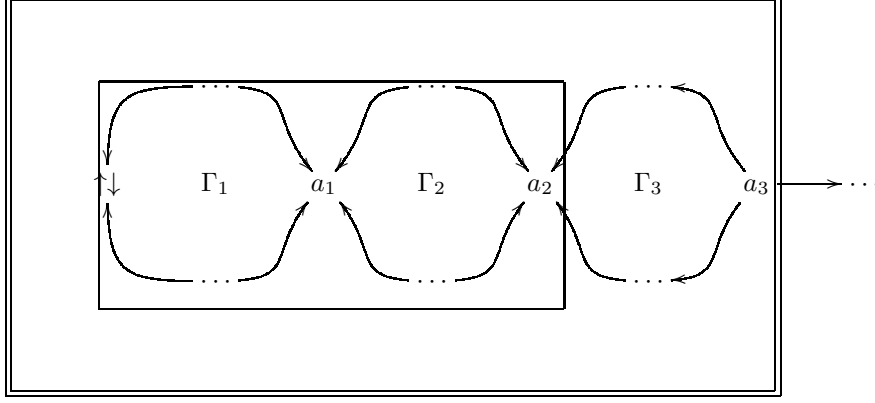


FIGURE 2.2. 2-chains as links in larger 2-chains

*Proof.* By grouping together links we see that it is enough to show the claim for a chain with two links (for short a “2-chain”). Indeed, if we can join two links  $\Gamma_1, \Gamma_2$  and have  $p$  injective we can say that this new chain  $\Gamma_1 \cup \Gamma_2$  is a link in a larger chain  $(\Gamma_1 \cup \Gamma_2) \cup \Gamma_3$  and continue as long as we have links left. Eventually we reach  $\Gamma = (\dots(\Gamma_1 \cup \Gamma_2) \cup \Gamma_3) \cup \dots \cup \Gamma_{n-1}) \cup \Gamma_n$ . Now

$$\begin{aligned} \tilde{p}[\mathbf{w}] &= (p^{(1)}[\mathbf{w}], p_n[\mathbf{w}]) = ((p^{(2)}[\mathbf{w}], p_{n-1}[\mathbf{w}]), p_n[\mathbf{w}]) = \dots \\ &= ((\dots((p_1[\mathbf{w}], p_2[\mathbf{w}]) \dots), p_{n-1}[\mathbf{w}]), p_n[\mathbf{w}]) \end{aligned}$$

be obtained from composition of the projections of each 2-chain. It is clear that  $\tilde{p}$  is injective iff  $p$  is injective.

Thus let  $\Gamma = \Gamma_1 \cup \Gamma_2$  with  $a$  the joining kink. We either have  $a \in p_1[\mathbf{w}] \vee a \in p_2[\mathbf{w}]$  or  $a \notin p_1[\mathbf{w}] \vee a \notin p_2[\mathbf{w}]$ . By letting  $\alpha = a$  if the first holds and  $\alpha = \varepsilon$  otherwise we get that there are words  $\mathbf{w}_1, \mathbf{w}_2$  such that  $\mathbf{w}_1 = \mathbf{x}_1 \alpha \mathbf{x}_2, \mathbf{w}_2 = \mathbf{y}_1 \alpha \mathbf{y}_2$  and  $[\mathbf{w}_1] = p_1[\mathbf{w}], [\mathbf{w}_2] = p_2[\mathbf{w}]$ , since  $a$  is a kink.

Now we define  $\mathbf{w} = \mathbf{x}_1 \mathbf{y}_1 \alpha \mathbf{x}_2 \mathbf{y}_2$ . Note that  $\mathbf{x}_1, \mathbf{x}_2$  commute with  $\mathbf{y}_1, \mathbf{y}_2$ . In fact this is welldefined as a function between elements since if  $\mathbf{x}_1 \alpha \mathbf{x}_2 = \tilde{\mathbf{x}}_1 \alpha \tilde{\mathbf{x}}_2$  and  $\mathbf{y}_1 \alpha \mathbf{y}_2 = \tilde{\mathbf{y}}_1 \alpha \tilde{\mathbf{y}}_2$  we have

$$\mathbf{x}_1 \mathbf{y}_1 \alpha \mathbf{x}_2 \mathbf{y}_2 = \mathbf{x}_1 \mathbf{y}_1 \alpha \mathbf{y}_2 \mathbf{x}_2 = \mathbf{x}_1 \tilde{\mathbf{y}}_1 \alpha \tilde{\mathbf{y}}_2 \mathbf{x}_2 = \tilde{\mathbf{y}}_1 \mathbf{x}_1 \alpha \mathbf{x}_2 \tilde{\mathbf{y}}_2 = \tilde{\mathbf{y}}_1 \tilde{\mathbf{x}}_1 \alpha \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_2 = \tilde{\mathbf{x}}_1 \tilde{\mathbf{y}}_1 \alpha \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_2,$$

so equivalent words give an equivalent word. Let us denote this function by  $\varphi : \text{Im}(p) \rightarrow HK_\Gamma$ .

Any element  $[\mathbf{w}'] \in HK_\Gamma$  has a word on the form  $\mathbf{w} = \mathbf{w}_1 \alpha \mathbf{w}_2$ , but since  $\Gamma \setminus \{a\}$  is disconnected each  $\mathbf{w}_i = \mathbf{x}_i \mathbf{y}_i, i \in \underline{2}$  where  $[\mathbf{x}_1], [\mathbf{x}_2] \in HK_{\Gamma'_1}$  and  $[\mathbf{y}_1], [\mathbf{y}_2] \in HK_{\Gamma'_2}$ .

$$\varphi \circ p([\mathbf{x}_1 \mathbf{y}_1 \alpha \mathbf{x}_2 \mathbf{y}_2]) = \varphi([\mathbf{x}_1 \alpha \mathbf{x}_2], [\mathbf{y}_1 \alpha \mathbf{y}_2]) = [\mathbf{x}_1 \mathbf{y}_1 \alpha \mathbf{x}_2 \mathbf{y}_2] \text{ and}$$

$p \circ \varphi([\mathbf{x}_1 \alpha \mathbf{x}_2], [\mathbf{y}_1 \alpha \mathbf{y}_2]) = p([\mathbf{x}_1 \mathbf{y}_1 \alpha \mathbf{x}_2 \mathbf{y}_2]) = ([\mathbf{x}_1 \alpha \mathbf{x}_2], [\mathbf{y}_1 \alpha \mathbf{y}_2])$  so  $\varphi$  and  $p$  are inverses of each other.  $\square$

**Corollary 21.** *If  $\Gamma$  is a chain and  $R_{f_i}$  is an effective representation of the monoid corresponding to each link, then  $R_f$  up induced from  $\{R_{f_i}\}$  is an effective representation of  $HK_\Gamma$ .*

*Proof.* Given  $R_f[\mathbf{x}]$  for some  $[\mathbf{x}] \in HK_\Gamma$  we can get  $R_{f_i} p_i[\mathbf{x}]$  by lemma 18. But since all  $R_{f_i}$  are effective this uniquely gives  $p_i[\mathbf{x}]$  for all  $i$ . Now apply  $\varphi$  from the previous theorem to obtain  $[\mathbf{x}]$ .  $\square$

*Remark 22.* Since the Hecke-Kiselman monoid of the graph  $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n$  is known (see remark 5) to have the effective representations  $R_f, f_{ij} \neq 0$  we get that any graph of type  $A_n$  has (by abuse of notation) the effective representations  $R_f, f_{ij} \neq 0$ . This answers the question 8 posed in [4]: “Is the representation  $(R_1)$  constructed above faithful (as semigroup representation)(when  $\Gamma$  is of type  $A_n$ )?” affirmatively. Here “faithful” means precisely effective, i.e. that the *semigroup representation* is injective.

**Corollary 23.** *Let  $\Gamma$  be a chain. Then  $\mathfrak{m}(\Gamma) = \Pi_{k=1}^n \mathfrak{m}(\Gamma_i)$ .*

*Proof.* The only property that separates  $\text{Im}(p)$  from general elements in  $HK_{\Gamma_1} \times HK_{\Gamma_2} \times \cdots \times HK_{\Gamma_n}$  is that kinks are simultaneously in or not in neighbouring elements. For elements with maximal content this restraint is void and we may choose elements independantly from the links. Thus we may use the multiplicative principle.  $\square$

Let  $\Gamma$  be of type  $A_n$  with  $k$  links and let  $l_i$  be the lenght of link  $i$ . By introducing a few help functions we can find a somewhat explicit formula for  $|HK_\Gamma|$ . Let  $\delta_i(A) = 1$  if  $i \in A$  and  $= 0$  otherwise for  $i \in \underline{k-1}$ . Let

$$c_i(A) = \begin{cases} C_{l_{i-1}}, & \text{if } \delta_{i-1}(A) = \delta_i(A) = 0 \\ C_{l_i} - C_{l_{i-1}}, & \text{if } \delta_{i-1}(A) + \delta_i(A) = 1 \\ C_{l_{i+1}} - 2C_{l_i} + C_{l_{i-1}}, & \text{if } \delta_{i-1}(A) = \delta_i(A) = 1 \end{cases}$$

for  $i \in \underline{k-1} \setminus \{1\}$ , let

$$c_1(A) = \begin{cases} C_{l_1}, & \text{if } \delta_1(A) = 0 \\ C_{l_{1+1}} - C_{l_1}, & \text{if } \delta_1(A) = 1 \end{cases}$$

and let

$$c_k(A) = \begin{cases} C_{l_k}, & \text{if } \delta_{k-1}(A) = 0 \\ C_{l_{k+1}} - C_{l_k}, & \text{if } \delta_{k-1}(A) = 1 \end{cases}$$

**Corollary 24.** *Let  $\Gamma$  be of type  $A_n$  and let  $l_i$  be the length of the  $i$ :th link. Then  $\mathfrak{m}(\Gamma) = \Pi_{i=1}^n C_{l_i}$  is a product of catalan numbers, and  $|HK_\Gamma| = \sum_{A \subseteq \underline{n}} (\prod_{i=1}^k c_i(A))$ .*

*Proof.* For the first claim, note that it suffices to know  $\mathfrak{m}(\Gamma_i)$  for every link by the previuos corollary. By [4] there is an isomorphism  $f$  between  $HK_{\Gamma_i}$  and the semigroup of order preserving and order decreasing transformations on the set  $\underline{l_i + 1}$ . In this setting  $f(v_i)(j) = j$  if  $j \neq i + 1$  and  $i$  otherwise. It follows that the image of an element of maximal content is a transformation that decreases all numbers (except 1) with at least 1.

On the other hand, any order preserving and order decreasing transformation which decreases all numbers (except 1) with at least 1 has to have maximal content since the only way to change any number  $i$  is to include  $v_{i-1}$  in the element. By a simple indexshift it is easy to see that

- (1) the number of order decreasing and order preserving transformations on  $\underline{l_k + 1}$  that decrease every number (except 1) with at least 1
  - (2) the number of order decreasing and order preserving transformations on  $\underline{l_k}$
- are equal and equal  $C_{l_k}$ .

For the second claim we need know how many elements in  $HK_{\Gamma_i}$  that have /do not have  $a_{i-1}, a_i$  respectively. We may start with the number of elements that have

neither  $a_{i-1}$  nor  $a_i$ . What remains is the Hecke-Kiselman monoid for a shorter graph, namely with the two leaves removed. By [4] there are  $C_{(l_i-2)+1} = C_{l_i-1}$  such elements. Similarly there are  $C_{l_i}$  elements in  $HK_{\Gamma_i}$  not containing  $a_i$ , and thus  $C_{l_i} - C_{l_i-1}$  elements containing exactly one of  $a_{i-1}$  or  $a_i$ . Since there are  $C_{l_i+1}$  elements in  $HK_{\Gamma_i}$ , there are  $C_{l_i+1} - 2(C_{l_i} - C_{l_i-1}) - C_{l_i-1} = C_{l_i+1} - 2C_{l_i} + C_{l_i-1}$  elements containing both  $a_{i-1}$  and  $a_i$ .  $\square$

**Theorem 25.** *Let  $\Gamma$  be a chain whose links are all on the form  $a \rightarrow b$ . Then  $|HK_{\Gamma}| = F_{2n+1}$  is the  $(2n+1)$ th Fibonacci number (where  $F_1 = F_2 = 1$ ).*

This series was guessed with help of The On-Line Encyclopedia of Integer Sequences [8], and was discovered independently by Grensing [2, 7].

*Proof.* Let  $f_n = |HK_{\Gamma_n}|$  where  $n = |V(\Gamma)|$ . It is clear that  $f_0 = |\{\varepsilon\}| = 1 = F_1$  and  $f_1 = |\{\varepsilon, a\}| = 2 = F_2$ . A well known equality of the Fibonacci numbers is

$$F_{i+4} - F_{i+2} = F_{i+3} = F_{i+2} + F_{i+1} = F_{i+2} + (F_{i+2} - F_i) = 2F_{i+2} - F_i.$$

This implies  $F_{2(n+2)+1} = 3F_{2(n+1)+1} - F_{2n+1}$ , i.e that the sequence  $(F_{2n+1})_{n \in \mathbb{N}}$  of odd Fibonacci numbers is uniquely determined by the recursive equality  $F_{i+4} - F_{i+2} = 2F_{i+2} - F_i$  and the values of  $F_1$  and  $F_3$ .

The number of elements in  $HK_{\Gamma_n}$  containing all vertices  $v_1, \dots, v_k$  but not  $v_{k+1}$  is  $2^k f_{n-k-1}$  for  $k \in \underline{n-1}$ . This leaves two extreme cases, namely elements containing all vertices and elements not containing the vertex  $v_1$ . Together these cases exhaust the possibilities and they are pairwise disjoint. This gives the following recursion equality

$$\begin{aligned} f_{n+1} &= f_n + \sum_{k=1}^{n-1} 2^k f_{n-k} + 2^{n+1} \iff \\ f_{n+1} - f_n &= \sum_{k=1}^{n-1} 2^k f_{n-k} + 2^{n+1} = 2 \sum_{k=1}^{n-1} 2^{k-1} f_{n-k} + 2^{n+1} = \\ &= 2 \sum_{k=0}^{n-2} 2^k f_{n-k-1} + 2^{n+1} = 2 \sum_{k=1}^{n-2} 2^k f_{(n-1)-k} + f_{n-1} + 2^{n+1} = \\ &= 2(f_{n-1} + \sum_{k=1}^{n-2} 2^k f_{(n-1)-k} + 2^n) - f_{n-1} = 2f_n - f_{n-1} \text{ which proves the claim. } \square \end{aligned}$$

**Definition 26.** An element  $[\mathbf{w}] \in HK_{\Gamma}$  is said to be *multiplicity free* if there exists a word  $\mathbf{w}' \in [\mathbf{w}]$  in which every vertex appears at most once. In other words  $[\mathbf{w}] \cap \mathcal{MF}_{V(\Gamma)} \neq \emptyset$ .

**Example 27.** Let  $\Gamma = a \rightarrow b \rightarrow c$ . Then  $[babc]$  is multiplicity free because  $abc \in [babc]$ . However,  $[bacb]$  is not multiplicity free since every word  $\mathbf{w} \in [bacb]$  contains at least two  $b$ s.

**Corollary 28.** *Let  $\Gamma$  be of type  $A_n$ . Then number of multiplicity free elements in  $HK_{\Gamma}$  is the Fibonacci number  $F_{2n+1}$ . Specifically, it does not depend on the orientation of the edges in  $\Gamma$ .*

*Proof.* We start by counting the number of multiplicity free maximal elements in  $\Gamma$ . It is easy to see that there are at least  $2^{n-1}$  such elements, since if  $a_k$  and  $a_{k+1}$  appear in different order in  $\mathbf{w}_1$  and  $\mathbf{w}_2$ , then  $\mathbf{w}_1 \not\sim \mathbf{w}_2$ .

If we can prove that there are no more than  $2^{n-1}$  multiplicity free maximal elements in  $\Gamma$  we may use the proof previous theorem to count the total number of multiplicity free elements.

We will prove that every multiplicity free element  $\mathbf{w} = a_{\pi(1)} a_{\pi(2)} \dots a_{\pi(n)} \sim a_{\tilde{\pi}(1)} a_{\tilde{\pi}(2)} \dots a_{\tilde{\pi}(n)}$ , where  $\tilde{\pi}$  satisfies

- (1)  $\tilde{\pi}(k) < \tilde{\pi}(k+1) \Rightarrow \tilde{\pi}(i) < \tilde{\pi}(k+1)$  for all  $i < k$ .
- (2)  $\tilde{\pi}(k) > \tilde{\pi}(k+1) \Rightarrow \tilde{\pi}(i) > \tilde{\pi}(k+1)$  for all  $i < k$ .

Since the orders of  $(\tilde{\pi}(k), \tilde{\pi}(k+1))$  determine  $\tilde{\pi}$  uniquely under the assumption on  $\tilde{\pi}$ , there can be no more than  $2^{n-1}$  such permutations.

We need to use substrings  $\mathbf{b}_i$  containing precisely the vertices  $a_j$  for  $j \leq i$ , which once formed are left untouched. Note that we have the following permitted permutations  $\mathbf{b}_i a_j = a_j \mathbf{b}_i$  for  $i < j + 1$ .

- (1) Set  $\mathbf{b}_1 = a_1$
- (2) Since  $\mathbf{b}_i$  commutes with every vertex not in  $\mathbf{b}_i$  *except*  $a_{i+1}$  we are free to move  $\mathbf{b}_i$  right next to  $a_{i+1}$ . If  $\mathbf{b}_i$  is to the left of  $a_{i+1}$ , set  $\mathbf{b}_{i+1} = \mathbf{b}_i a_{i+1}$ , otherwise set  $\mathbf{b}_{i+1} = a_{i+1} \mathbf{b}_i$

It is clear that the word we end up with equals  $\mathbf{w}$  and has the desired permutation, by construction.  $\square$

**Example 29.** We illustrate the procedure above by showing that the word  $a_2 a_3 a_1 a_5 a_4$  equals the word  $a_5 a_2 a_1 a_3 a_4$  for any orientation on the edges in  $a_1 - a_2 - a_3 - a_4 - a_5$ .

$$(2.1) \quad a_2 a_3 a_1 a_5 a_4$$

$$(2.2) \quad (\mathbf{b}_1 = a_1) \sim a_2 a_1 a_3 a_5 a_4$$

$$(2.3) \quad (\mathbf{b}_2 = a_2 a_1) \sim a_2 a_1 a_3 a_5 a_4$$

$$(2.4) \quad (\mathbf{b}_3 = a_2 a_1 a_3) \sim a_5 a_2 a_1 a_3 a_4$$

$$(2.5) \quad (\mathbf{b}_4 = a_2 a_1 a_3 a_4) \sim a_5 a_2 a_1 a_3 a_4$$

If we identify the first and last vertices in a path we get a cycle. Similarly, if we join the first and last link in a chain in the same way every other link is joined, namely that their intersection is a kink, we get a closed chain which we will discuss later. In order to do this we need a more explicit construction of the inverse function  $p^{-1} : \text{Im}(p) \rightarrow HK_\Gamma$  where at least three links are involved. In fact, we only need to describe the construction for a chain with three links. Indeed, we can view a series of links as a link itself, as we will see later.

**Definition 30.** A *block*  $A_i = \mathbf{x}_1 \mathbf{y}_1 \alpha_i \mathbf{x}_2 \mathbf{y}_2$  is a word in which

- (1)  $[\mathbf{x}_1], [\mathbf{x}_2] \in HK_{\Gamma'_{i-1}}$
- (2)  $[\mathbf{y}_1], [\mathbf{y}_2] \in HK_{\Gamma'_i}$
- (3)  $\alpha_i$  is  $a_i$  or the empty word  $\varepsilon$

*Remark 31.* We are suppressing notation for readability. Of course  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1$  and  $\mathbf{y}_2$  also depend on  $i$ .

**Lemma 32.** Let  $\Gamma$  be a chain with three links. Then every element  $[\mathbf{w}'] \in HK_\Gamma$  has a word  $\mathbf{w}$  on the form  $\mathbf{w} = A_{\pi(1)} A_{\pi(2)}$  where  $A_1, A_2$  are blocks and  $\pi$  is a permutation on  $\underline{2}$ .

*Proof.* By lemma 12 we may take a  $\mathbf{w}$  which is multiplicity free with respect to the kinks  $a_1, a_2$ . If one of  $a_1, a_2$  does not appear in  $\mathbf{w}$ , we insert  $\alpha_1$  or  $\alpha_2$  somewhere. Otherwise we change  $a_1, a_2$  to  $\alpha_1, \alpha_2$  respectively and get  $\mathbf{w} = \mathbf{w}_1 \alpha_{\pi(1)} \mathbf{w}_2 \alpha_2 \mathbf{w}_3$ . Since  $\Gamma \setminus \{a_1, a_2\}$  has three components which are pairwise disjoint each  $\mathbf{w}_i$  can be assumed to be on the form  $\mathbf{w}_i = \mathbf{x}_i \mathbf{y}_i \mathbf{z}_i$ , where  $[\mathbf{x}_i] \in HK_{\Gamma_1 \setminus \{a_1\}}, [\mathbf{y}_i] \in HK_{\Gamma_2 \setminus \{a_1, a_2\}}$  and  $[\mathbf{z}_i] \in HK_{\Gamma_3 \setminus \{a_2\}}$ . Thus  $\mathbf{w} = \mathbf{x}_1 \mathbf{y}_1 \mathbf{z}_1 \alpha_{\pi(1)} \mathbf{x}_2 \mathbf{y}_2 \mathbf{z}_2 \alpha_{\pi(2)} \mathbf{x}_3 \mathbf{y}_3 \mathbf{z}_3$ . Note that a lot of elements commute, namely  $\mathbf{x}_i \mathbf{y}_j = \mathbf{y}_j \mathbf{x}_i, \mathbf{x}_i \mathbf{z}_j = \mathbf{z}_j \mathbf{x}_i, \mathbf{y}_i \mathbf{z}_j = \mathbf{z}_j \mathbf{y}_i$  and  $\mathbf{x}_i \alpha_2 = \alpha_2 \mathbf{x}_i, \mathbf{z}_i \alpha_1 = \alpha_1 \mathbf{z}_i$  for all  $i, j \in \underline{3}$ . Assume  $\pi = \text{id}$ . Then

$$\mathbf{w} = \mathbf{x}_1 \mathbf{y}_1 \mathbf{z}_1 \alpha_1 \mathbf{x}_2 \mathbf{y}_2 \mathbf{z}_2 \alpha_2 \mathbf{x}_3 \mathbf{y}_3 \mathbf{z}_3 \sim (\mathbf{x}_1 \mathbf{y}_1 \alpha_1 \mathbf{x}_1 \mathbf{x}_3 \mathbf{y}_2)(\mathbf{z}_1 \mathbf{z}_3 \alpha_2 \mathbf{y}_3 \mathbf{z}_3)$$

Similarly if  $\pi \neq \text{id}$  we get

$$\mathbf{w} = \mathbf{x}_1 \mathbf{y}_1 \mathbf{z}_1 \alpha_2 \mathbf{x}_2 \mathbf{y}_2 \mathbf{z}_2 \alpha_1 \mathbf{x}_3 \mathbf{y}_3 \mathbf{z}_3 \sim (\mathbf{y}_1 \mathbf{z}_1 \alpha_2 \mathbf{y}_2 \mathbf{z}_2 \mathbf{z}_3)(\mathbf{x}_1 \mathbf{x}_2 \alpha_1 \mathbf{x}_3 \mathbf{y}_3).$$

We have the desired block formes.  $\square$

Given  $[\mathbf{w}_1] \in HK_{\Gamma_1}$ ,  $[\mathbf{w}_2] \in HK_{\Gamma_2}$  and  $[\mathbf{w}_3] \in HK_{\Gamma_3}$  we may do the following.

- (1) Choose a word  $\mathbf{w}'_i$  which is multiplicity free with respect to the kinks.
- (2) For  $\mathbf{w}_2 = (\mathbf{y}_1 \alpha_{\pi(1)} \mathbf{y}_2)(\alpha_{\pi(2)} \mathbf{y}_3)$  we choose one subword for  $\alpha_1$  and one for  $\alpha_2$ . For the others we have  $\mathbf{w}_1 = \mathbf{x}_1 \alpha_1 \mathbf{x}_2$  and  $\mathbf{w}_3 = \mathbf{z}_1 \alpha_2 \mathbf{z}_2$
- (3) We build the blocks  $A_1 = \mathbf{x}_1 \tilde{\mathbf{y}}_1 \alpha_1 \mathbf{x}_2 \tilde{\mathbf{y}}_2$  and  $A_2 = \tilde{\mathbf{y}}_3 \mathbf{z}_1 \alpha_2 \tilde{\mathbf{y}}_4 \mathbf{z}_2$  where

$$(\tilde{\mathbf{y}}_1, \tilde{\mathbf{y}}_2, \tilde{\mathbf{y}}_3, \tilde{\mathbf{y}}_4) = \begin{cases} (\mathbf{y}_1, \mathbf{y}_2, \varepsilon, \mathbf{y}_3), & \pi = \text{id} \\ (\varepsilon, \mathbf{y}_3, \mathbf{y}_1, \mathbf{y}_2) & \pi \neq \text{id} \end{cases}$$

- (4) Set  $\mathbf{w} = A_{\pi(1)} A_{\pi(2)}$  where  $\pi$  is the same permutation as in  $\mathbf{w}_2$ .

**Proposition 33.** *The construction above is independent of the choice of words  $\mathbf{w}'_i \in [\mathbf{w}_i]$ , i.e.  $[\mathbf{w}_1], [\mathbf{w}_2], [\mathbf{w}_3]$  uniquely determine  $[\mathbf{w}]$ . Moreover  $p[\mathbf{w}] = ([\mathbf{w}_1], [\mathbf{w}_2], [\mathbf{w}_3])$ .*

*Proof.* Assume  $\mathbf{x}_1 \alpha_1 \mathbf{x}_2 \sim \tilde{\mathbf{x}}_1 \alpha_1 \tilde{\mathbf{x}}_2$ ,  $\mathbf{y}_1 \alpha_{\pi(1)} \mathbf{y}_2 \alpha_{\pi(2)} \mathbf{y}_3 \sim \tilde{\mathbf{y}}_1 \alpha_{\tilde{\pi}(1)} \tilde{\mathbf{y}}_2 \alpha_{\tilde{\pi}(2)} \tilde{\mathbf{y}}_3$  and  $\mathbf{z}_1 \alpha_2 \mathbf{z}_2 \sim \tilde{\mathbf{z}}_1 \alpha_2 \tilde{\mathbf{z}}_2$ . We will associate anything with a tilde with the construction from  $[\tilde{\mathbf{w}}_1], [\tilde{\mathbf{w}}_2], [\tilde{\mathbf{w}}_3]$ . For symmetry reasons we may assume  $\pi = \text{id}$ . Then  $\mathbf{w} = A_1 A_2 = \mathbf{x}_1 \mathbf{y}_1 \alpha_1 \mathbf{x}_2 \mathbf{y}_2 \mathbf{z}_1 \alpha_2 \mathbf{y}_3 \mathbf{z}_2$ . Now we need to separate into cases depending on if  $\tilde{\mathbf{w}}_2 = \tilde{\mathbf{y}}_1 \alpha_1 \tilde{\mathbf{y}}_2 \alpha_2 \tilde{\mathbf{y}}_3$  or  $\tilde{\mathbf{w}}_2 = \tilde{\mathbf{y}}_1 \alpha_2 \tilde{\mathbf{y}}_2 \alpha_1 \tilde{\mathbf{y}}_3$

- (1)  $\tilde{\pi} = \text{id}$ :

$$\begin{aligned} \mathbf{w} = A_1 A_2 &= \mathbf{x}_1 \mathbf{y}_1 \alpha_1 \mathbf{x}_2 \mathbf{y}_2 \mathbf{z}_1 \alpha_2 \mathbf{y}_3 \mathbf{z}_2 \sim \mathbf{x}_1 \mathbf{z}_1 (\mathbf{y}_1 \alpha_1 \mathbf{y}_2 \alpha_2 \mathbf{y}_3) \mathbf{x}_2 \mathbf{z}_2 \sim \\ &\mathbf{x}_1 \mathbf{z}_1 (\tilde{\mathbf{y}}_1 \alpha_1 \tilde{\mathbf{y}}_2 \alpha_2 \tilde{\mathbf{y}}_3) \mathbf{x}_2 \mathbf{z}_2 \sim \tilde{\mathbf{y}}_1 (\mathbf{x}_1 \alpha_1 \mathbf{x}_2) \tilde{\mathbf{y}}_2 (\mathbf{z}_1 \alpha_2 \mathbf{z}_2) \tilde{\mathbf{y}}_3 \sim \\ &\tilde{\mathbf{y}}_1 (\tilde{\mathbf{x}}_1 \alpha_1 \tilde{\mathbf{x}}_2) \tilde{\mathbf{y}}_2 (\tilde{\mathbf{z}}_1 \alpha_2 \tilde{\mathbf{z}}_2) \tilde{\mathbf{y}}_3 \sim \tilde{\mathbf{x}}_1 \tilde{\mathbf{y}}_1 \alpha_1 \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_2 \tilde{\mathbf{z}}_1 \alpha_2 \tilde{\mathbf{y}}_3 \tilde{\mathbf{z}}_2 = \tilde{A}_1 \tilde{A}_2 = \tilde{\mathbf{w}} \end{aligned}$$

- (2)  $\tilde{\pi} \neq \text{id}$ :

$$\begin{aligned} \mathbf{w} = A_1 A_2 &= \mathbf{x}_1 \mathbf{y}_1 \alpha_1 \mathbf{x}_2 \mathbf{y}_2 \mathbf{z}_1 \alpha_2 \mathbf{y}_3 \mathbf{z}_2 \sim \mathbf{y}_1 (\mathbf{x}_1 \alpha_1 \mathbf{x}_2) \mathbf{y}_2 (\mathbf{z}_1 \alpha_2 \mathbf{z}_2) \mathbf{y}_3 \sim \\ &\mathbf{y}_1 (\tilde{\mathbf{x}}_1 \alpha_1 \tilde{\mathbf{x}}_2) \mathbf{y}_2 (\tilde{\mathbf{z}}_1 \alpha_2 \tilde{\mathbf{z}}_2) \mathbf{y}_3 \sim \tilde{\mathbf{x}}_1 \tilde{\mathbf{z}}_1 (\mathbf{y}_1 \alpha_1 \mathbf{y}_2 \alpha_2 \mathbf{y}_3) \tilde{\mathbf{x}}_2 \tilde{\mathbf{z}}_2 \sim \\ &\tilde{\mathbf{x}}_1 \tilde{\mathbf{z}}_1 (\tilde{\mathbf{y}}_1 \alpha_2 \tilde{\mathbf{y}}_2 \alpha_1 \tilde{\mathbf{y}}_3) \tilde{\mathbf{x}}_2 \tilde{\mathbf{z}}_2 \sim \tilde{\mathbf{y}}_1 \tilde{\mathbf{z}}_1 \alpha_2 \tilde{\mathbf{y}}_2 \tilde{\mathbf{z}}_2 \tilde{\mathbf{x}}_1 \alpha_1 \tilde{\mathbf{y}}_3 \tilde{\mathbf{x}}_2 = \tilde{A}_2 \tilde{A}_1 = \tilde{\mathbf{w}} \end{aligned}$$

But

$$p[\mathbf{x}_1 \mathbf{y}_1 \alpha_1 \mathbf{x}_2 \mathbf{y}_2 \mathbf{z}_1 \alpha_2 \mathbf{y}_3 \mathbf{z}_2] = ([\mathbf{x}_1 \alpha_1 \mathbf{x}_2], [\mathbf{y}_1 \alpha_1 \mathbf{y}_2 \alpha_2 \mathbf{y}_3], [\mathbf{z}_1 \alpha_2 \mathbf{z}_2])$$

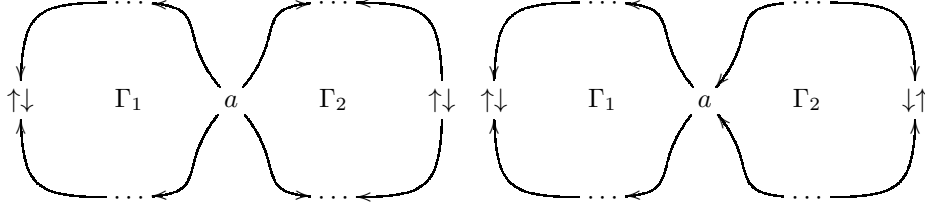
and

$$p[\mathbf{y}_1 \mathbf{z}_1 \alpha_2 \mathbf{y}_2 \mathbf{z}_2 \mathbf{x}_1 \alpha_2 \mathbf{x}_2 \mathbf{y}_3] = ([\mathbf{x}_1 \alpha_1 \mathbf{x}_2], [\mathbf{y}_1 \alpha_2 \mathbf{y}_2 \alpha_1 \mathbf{y}_3], [\mathbf{z}_1 \alpha_2 \mathbf{z}_2])$$

so  $p[\mathbf{w}] = ([\mathbf{w}_1], [\mathbf{w}_2], [\mathbf{w}_3])$ .  $\square$

### 3. CARDINALITIES AND EFFECTIVENESS

**Theorem 34.** *Let  $\Gamma$  be a chain with two links and let  $\tilde{\Gamma}$  be obtained by switching orientation on all edges in  $\Gamma_2$ . Then  $|HK_{\Gamma}| \leq |HK_{\tilde{\Gamma}}|$  with equality iff  $a$  has no adjacent vertex in at least one of  $\Gamma_1, \Gamma_2$ .*

FIGURE 3.1.  $\Gamma$  and  $\tilde{\Gamma}$ 

*Proof.* Since  $a \in \Gamma$  is a kink any element  $[\mathbf{w}'] \in HK_\Gamma$  has a word on the form  $\mathbf{w} = \mathbf{x}_1 \mathbf{y}_1 \alpha \mathbf{x}_2 \mathbf{y}_2$ , where as before  $\alpha$  is  $a$  or  $\varepsilon$  and  $[\mathbf{x}_1], [\mathbf{x}_2] \in HK_{\Gamma'_1}, [\mathbf{y}_1], [\mathbf{y}_2] \in HK_{\Gamma'_2}$ . From this we may construct  $\tilde{\mathbf{w}} = \mathbf{x}_1 \overleftarrow{\mathbf{y}}_2 \alpha \overleftarrow{\mathbf{y}}_1 \mathbf{x}_2$ , where  $\overleftarrow{\mathbf{y}}$  is the reverse of  $\mathbf{y}$ , as a word. We notice that this is a welldefined function  $f : HK_\Gamma \rightarrow HK_{\tilde{\Gamma}}$  since if  $\mathbf{x}_1 \mathbf{y}_1 \alpha \mathbf{x}_2 \mathbf{y}_2 \sim \tilde{\mathbf{x}}_1 \tilde{\mathbf{y}}_1 \alpha \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_2$  we get, by theorem 20 that  $\mathbf{x}_1 \alpha \mathbf{x}_2 \sim \tilde{\mathbf{x}}_1 \alpha \tilde{\mathbf{x}}_2$  and  $\mathbf{y}_1 \alpha \mathbf{y}_2 \sim \tilde{\mathbf{y}}_1 \alpha \tilde{\mathbf{y}}_2$ . By reversing the strings in the latter equality we get  $\overleftarrow{\mathbf{y}}_2 \alpha \overleftarrow{\mathbf{y}}_1 \sim \overleftarrow{\tilde{\mathbf{y}}_2} \alpha \overleftarrow{\tilde{\mathbf{y}}_1}$ . Thus

$$\begin{aligned} f(\mathbf{x}_1 \mathbf{y}_1 \alpha \mathbf{x}_2 \mathbf{y}_2) &= \mathbf{x}_1 \overleftarrow{\mathbf{y}}_2 \alpha \overleftarrow{\mathbf{y}}_1 \mathbf{x}_2 \sim \mathbf{x}_1 \overleftarrow{\mathbf{y}}_2 \alpha \overleftarrow{\mathbf{y}}_1 \mathbf{x}_2 \sim \overleftarrow{\mathbf{y}}_2 \mathbf{x}_1 \alpha \mathbf{x}_2 \overleftarrow{\mathbf{y}}_1 \sim \\ &\quad \overleftarrow{\mathbf{y}}_2 \tilde{\mathbf{x}}_1 \alpha \tilde{\mathbf{x}}_2 \overleftarrow{\mathbf{y}}_1 \sim \tilde{\mathbf{x}}_1 \overleftarrow{\mathbf{y}}_2 \alpha \overleftarrow{\mathbf{y}}_1 \tilde{\mathbf{x}}_2 = f(\tilde{\mathbf{x}}_1 \tilde{\mathbf{y}}_1 \alpha \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_2). \end{aligned}$$

Notice that the above argument works as well in the other direction for elements  $[\tilde{\mathbf{w}}] \in HK_{\tilde{\Gamma}}$ . Thus  $|HK_\Gamma| \leq |HK_{\tilde{\Gamma}}|$ . But if there is at least one vertex adjacent to  $a$  in each link we have the following subgraph  $x \rightarrow a \rightarrow y$  in  $\tilde{\Gamma}$  which means that the element  $[axya] \in HK_{\tilde{\Gamma}}$ .

If adjacent vertices are missing in one of the links it follows that  $\Gamma$  is disconnected. If  $\Gamma_1$  is disconnected we have that

$$|HK_\Gamma| = |HK_{\Gamma_1 \setminus \{a\}}| |HK_{\Gamma_2}| = |HK_{\Gamma_1 \setminus \{a\}}| |HK_{\tilde{\Gamma}_2}| = |HK_{\tilde{\Gamma}}|.$$

Otherwise  $\Gamma_2$  is disconnected and we have that

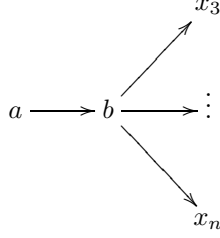
$$|HK_\Gamma| = |HK_{\Gamma_1}| |HK_{\Gamma_2 \setminus \{a\}}| = |HK_{\Gamma_1}| |HK_{\tilde{\Gamma}_2 \setminus \{a\}}| = |HK_{\tilde{\Gamma}}|.$$

□

**Corollary 35.** *Let  $\Gamma$  be of type  $A_n$ . Then  $F_{2n+1} \leq |HK_\Gamma| \leq C_{n+1}$  with equality on the left iff every vertex of  $\Gamma$  is a kink and equality on the right iff  $\Gamma \simeq a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n$ .*

*Proof.* The right inequality (including conditions) was proved in [4]. We proved in corollary 28 that  $\Gamma$  of type  $A_n$  with only kinks has  $F_{2n+1}$  elements. Using the previous theorem we see that removing kinks (by change of directions) results in a larger monoid. □

Consider the graph  $\Gamma$  with  $V(\Gamma) = \{a, b, x_3, \dots, x_n\}$  and  $E(\Gamma) = \{(a \rightarrow b)\} \cup \{(b \rightarrow x_i) | i \in \underline{n} \setminus \{1, 2\}\}$



We will call a graph like this a  $(1, k)$ -star, where  $k = n - 2$  is the number of sinks.

**Lemma 36.** *For a  $(1, k)$ -star as above  $Max_\Gamma = \{[xbyabz]\} \sqcup \{[x'bay']\}$  such that  $\mathbf{xyz}$  is a word containing each vertex  $x_i$  exactly once and  $\mathbf{x'y'}$  is a word containing each vertex  $x_i$  exactly once. Moreover, each choice of  $([\mathbf{x}], [\mathbf{y}], [\mathbf{z}])$  or  $([\mathbf{x'}], [\mathbf{y'}])$  gives a unique element.*

*Proof.* Since  $a$  is a kink any element  $[\mathbf{w}'] \in M$  has a word  $\mathbf{w} = \mathbf{w}_1 a \mathbf{w}_2$  where  $\mathbf{w}_1, \mathbf{w}_2 \in HK_{\Gamma \setminus \{a\}}$ . Also,  $b$  is a kink of  $\Gamma \setminus \{a\}$ , so each  $\mathbf{w}_1, \mathbf{w}_2$  can be assumed to have at most one  $b$ . This leads to one of the following cases

- (1)  $\mathbf{w} = \mathbf{x}_1 b \mathbf{x}_2 a \mathbf{x}_3 b \mathbf{x}_4 \sim \mathbf{x}_1 b (\mathbf{x}_2 \mathbf{x}_3) a b \mathbf{x}_4$
- (2)  $\mathbf{w} = \mathbf{x}_1 a \mathbf{x}_2 b \mathbf{x}_3 \sim (\mathbf{x}_1 \mathbf{x}_2) b a b \mathbf{x}_3$
- (3)  $\mathbf{w} = \mathbf{x}_1 b \mathbf{x}_2 a \mathbf{x}_3 \sim \mathbf{x}_1 b a (\mathbf{x}_2 \mathbf{x}_3)$

where  $X_i \in \Gamma \setminus \{a, b\}$ . The case where  $b$  is missing from both sides is excluded since we are dealing only with elements of maximal content.

Since each  $x_i$  is a kink in  $\Gamma$  we may assume that  $\mathbf{x}_1 \mathbf{x}_2 \mathbf{x}_3 (\mathbf{x}_4)$  is a word containing each vertex  $x_i$  exactly once.

If we project onto the subgraph  $a \rightarrow b$  we find that the first two cases project to  $[ab]$  and the third case to  $[ba]$ . We may join the first two as  $\mathbf{w} = \mathbf{xbyabz}$ .

Note that  $\Gamma \setminus \{a, b\}$  is completely disconnected, so each  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  is determined completely by it's content. Each choice of where  $x_i$  goes gives a different element as can be seen by projection to the subgraph  $a \rightarrow b \rightarrow x_i$ . We get  $[x_i ab], [ba x_i b], [ab x_i]$  for cases 1 and 2, which are clearly different. In the third case we get  $[x_i ba], [ba x_i]$  which, too, clearly are different.  $\square$

**Corollary 37.** *Let  $\Gamma$  be a  $(1, k)$ -star as above. Then  $\mathfrak{m}(\Gamma) = 3^{n-2} + 2^{n-2}$ .*

*Proof.* In the case  $w = \mathbf{xbyabz}$  each vertex  $x_i$  can be chosen independently to be in  $\mathbf{x}, \mathbf{y}$  or  $\mathbf{z}$ . This gives  $3^{n-2}$  different elements of maximal content. The other case gives  $2^{n-1}$  as each  $x_i$  can go independently to  $\mathbf{x'}$  or  $\mathbf{y'}$ .  $\square$

**Theorem 38.** *Let  $\Gamma$  be a  $(1, k)$ -star as above. Then  $R_1$  is effective.*

*Proof.* We only need to take care of elements of maximal content since any proper subgraph is either

- (1) a smaller graph of the same type,
- (2) totally disconnected or
- (3) a tree with only kinks, which falls into the chain case.

Since we only study elements of maximal content no vertex is left untouched by representation matrices. This leaves three possibilities for each  $x_i$  and we will use them to build  $\mathbf{x}, \mathbf{y}$  and  $\mathbf{z}$ .

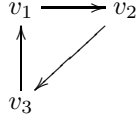
- (1) If  $R_1[\mathbf{w}](x_i) = b$  then we put  $x_i$  in  $\mathbf{x}$ ,



- (2) if  $R_1[\mathbf{w}](x_i) = a$  then we put  $x_i$  in  $\mathbf{y}$ ,
- (3) otherwise  $R_1[\mathbf{w}](x_i) = 0$  and we put  $x_i$  in  $\mathbf{z}$ .

If at least one  $R_1[\mathbf{w}](x_i) = 0$  or  $R_1[\mathbf{w}](b) = 0$  we are in the case  $\mathbf{w} = \mathbf{xbyabz}$  and we have defined a unique element  $[\mathbf{w}]$  from  $R_1[\mathbf{w}]$ . Otherwise  $R_1[\mathbf{w}](b) = a$ , which means that we have the case  $\mathbf{w} = \mathbf{xaby}$ . Again we have defined a unique element  $[\mathbf{w}]$  from  $R_1[\mathbf{w}]$ .  $\square$

**Theorem 39.**  $R_2$  is an effective representation of  $HK_{C_3}$  for the directed 3-cycle  $C_3$  oriented as below.



*Proof.* Recall that we use the cyclic indexing  $i \in \mathbb{Z}_3$ . There are two types of elements in  $HK_{C_3}$ , namely  $[v_i v_{i+1} v_{i+2} \cdots v_j]$  (increasing) and  $[v_i v_{i-1} v_{i-2} \cdots v_j]$  (decreasing) and they are disjoint apart from the elements  $[\varepsilon], [v_1], [v_2]$  and  $[v_3]$ . Indeed, if we try to make an irreducible word  $\mathbf{w}$  longer by attaching one new vertex to the right we can not use either of the previous two. Since  $C_3$  has only three vertices, there is only one choice if  $\mathbf{w}$  contains at least two letters. On the other hand we can use the representation  $R_2$  to prove that all such elements are in fact distinct. Thus all we need to know to uniquely identify an element is its first letter, if it increases or decreases and the length. We may use the following algorithm. We assume that  $\mathbf{w}$  is of minimal length.

- If there exists  $i \in \mathbb{Z}_3$  such that  $R_2[\mathbf{w}](v_i) = v_i$ 
  - $\mathbf{w} = \varepsilon, v_j, v_j v_{j+1}$  or  $v_j v_{j-1}$  for some  $j \in \mathbb{Z}_3$ , each of which has a unique representation. End algorithm.
- If there is only one vertex  $v_i$  in the image ( $R_2[\mathbf{w}](v_j) = 2^k v_i$  for all  $j$ )
  - $\mathbf{w}$  is increasing. The length equals the highest exponent and the first vertex is  $v_{i-1}$ . End algorithm.
- Otherwise  $\mathbf{w}$  is decreasing. There will be exactly two vertices  $v_i, v_{i+1}$  in the image. The first vertex in  $\mathbf{w}$  is  $v_{i+2}$ . Let  $k$  be the highest exponent.
- If there is exactly one vertex with the highest exponent
  - The length of  $\mathbf{w}$  is  $l = 2k - 1$ .
- Otherwise the length of  $\mathbf{w}$  is  $l = 2k$ .

$\square$

**Proposition 40.** Let  $C_n$  be the directed  $n$ -cycle. If there is a function  $f: E(C_n) \rightarrow \mathcal{R}$  such that  $R_f$  is effective, then  $R_2: HK_{C_n} \rightarrow \text{End}_{\mathbb{Z}}(\bigoplus_{v \in V(C_n)} \mathbb{Z}v)$  is effective.

*Proof.* Recall that we use cyclic indexing for oriented cycles. Every vertex  $v_i$  has only one other vertex pointing to it. Thus  $R_f[\mathbf{w}](v_i) = \prod_{k=j}^{i-1} f_{k(k+1)} v_j$  for some vertex  $v_j$ , which means that knowing the length of the path  $v_j \rightarrow \cdots \rightarrow v_i$  (possibly with multiple passes through the same vertices) is enough to recover  $[\mathbf{w}]$ . But the length of the path equals the exponent  $k$  in  $R_2[\mathbf{w}](v_i) = 2^k v_j$ .  $\square$

#### 4. PARTIAL RESULTS

This section contains results that either need conjecture 42 in order to be meaningful, or would be significantly stronger if it turned out to be true.

**Definition 41.** Let  $a, b \in \Gamma$  be kinks. If for every pair of  $\mathbf{w} \sim \mathbf{w}'$  such that  $a$  and  $b$  appear exactly once (each) in  $\mathbf{w}$  and  $\mathbf{w}'$  and come in the same order there is a series of words  $\mathbf{w} = \mathbf{w}_1 \approx \mathbf{w}_2 \approx \dots \approx \mathbf{w}_k = \tilde{\mathbf{w}}$  such that  $a$  and  $b$  are multiplicity free in each  $\mathbf{w}_i$  and the relation  $ab = ba$  is not used we say that the triple  $(a, b, \Gamma)$  is *stable*.

**Conjecture 42.** Every pair of kinks  $a, b \in \Gamma$  gives a stable triple  $(a, b, \Gamma)$ .

We start by proving that a large number of triples are stable.

**Proposition 43.** Let  $a, b \in \Gamma$  be kinks. If either of the following are true, then  $(a, b, \Gamma)$  is stable.

- (1)  $a$  and  $b$  are adjacent. Notably the graph for the Kiselman monoid  $K_n$  satisfies this with  $a$  and  $b$  as the (unique) source and sink.
- (2) For any vertex  $x \in \Gamma$  we have that  $a \rightarrow x \iff b \rightarrow x$  and  $x \rightarrow a \iff x \rightarrow b$ . In other words  $\Gamma$  has a graph automorphism that switches the vertices  $a$  and  $b$  and leaves all other vertices untouched.
- (3)  $a$  and  $b$  are in separate connected components.
- (4)  $\Gamma = \Gamma_1 \cup \Gamma_2$  is a chain with two links  $\Gamma_1, \Gamma_2$  and  $a \in \Gamma_1 \setminus \Gamma_2, b \in \Gamma_2 \setminus \Gamma_1$  are kinks. Moreover, for  $c \in \Gamma_1 \cap \Gamma_2$  we have that  $(a, c, \Gamma_1)$  and  $(b, c, \Gamma_2)$  are stable.
- (5)  $\Gamma = \Gamma_1 \cup \Gamma_2$  such that  $V(\Gamma_1 \cap \Gamma_2) = \{a, b\}$ . Moreover we require that  $(a, b, \Gamma_1)$  and  $(a, b, \Gamma_2)$  are stable.

*Proof.* W.l.o.g, assume  $a, b \in c(\mathbf{w})$  and  $a$  comes before  $b$  in  $\mathbf{w}$  and  $\mathbf{w}'$ . By abuse of notation, let  $p_i : (V(\Gamma))^* \rightarrow (V(\Gamma_i))^*$  be the projection which eliminates vertices not in  $V(\Gamma_i)$ . Note that  $\mathbf{w} \sim \mathbf{w}' \Rightarrow p_i(\mathbf{w}) \sim p_i(\mathbf{w}')$ .

- (1) Obvious.
- (2) Assume that we have a sequence of words  $\mathbf{w} = \mathbf{w}_1 \approx \mathbf{w}_2 \approx \dots \approx \mathbf{w}_k = \mathbf{w}'$  where  $\mathbf{w}_i \in \mathcal{MF}_{\{a,b\}}$ . Consider the function  $\psi : \mathcal{MF}_{\{a,b\}} \rightarrow \mathcal{MF}_{\{a,b\}}$  defined by  $\psi(\mathbf{x}_1 a \mathbf{x}_2 b \mathbf{x}_3) = \mathbf{x}_1 a \mathbf{x}_2 b \mathbf{x}_3$  and  $\psi(\mathbf{x}_1 b \mathbf{x}_2 a \mathbf{x}_3) = \mathbf{x}_1 a \mathbf{x}_2 b \mathbf{x}_3$ . Note that  $\psi(\mathbf{w}) = \mathbf{w}$  and  $\psi(\mathbf{w}') = \mathbf{w}'$ . We have to separate into cases depending on which edge relation is used.
  - (a) Relations  $x^2 = x, xyx = yxy = xy$  for  $x, y \neq a, b$ . We immediately get  $\mathbf{w}_i \approx \mathbf{w}_{i+1} \Rightarrow \psi(\mathbf{w}_i) \approx \psi(\mathbf{w}_{i+1})$
  - (b) Relations  $ax = xax, ya = yay, bx = xbx, yb = yby$  for  $x, y \neq a, b$ . The same relation can be used, but with  $a$  instead of  $b$  and vice versa if needed. Thus  $\mathbf{w}_i \approx \mathbf{w}_{i+1} \Rightarrow \psi(\mathbf{w}_i) \approx \psi(\mathbf{w}_{i+1})$ .
  - (c) Relation  $ab = ba$ . We get that  $\mathbf{w}_i \approx \mathbf{w}_{i+1} \Rightarrow \psi(\mathbf{w}_i) = \psi(\mathbf{w}_{i+1}) \Rightarrow \psi(\mathbf{w}_i) \approx \psi(\mathbf{w}_{i+1})$   
Put together,  $\psi(\mathbf{w}_i)$  gives a desired series of words between  $\mathbf{w}$  and  $\mathbf{w}'$ .
- (3) We have  $\mathbf{w} \sim \mathbf{x}_1 a \mathbf{x}_2 \mathbf{y}_1 b \mathbf{y}_2 = p_1(\mathbf{w}) p_2(\mathbf{w}) \sim p_1(\mathbf{w}') p_2(\mathbf{w}') \sim \mathbf{x}'_1 a \mathbf{x}'_2 \mathbf{y}'_1 b \mathbf{y}'_2 \sim \mathbf{w}'$
- (4) If  $c \notin c(\mathbf{w})$  we fall into the third case. W.l.o.g. assume that  $c$  appears in  $\mathbf{w}$  exactly once. By symmetry we may assume that  $c$  comes before  $b$  in  $\mathbf{w}$ . This gives six cases, since  $\mathbf{w}$  can have  $c$  before or after  $a$  and  $\mathbf{w}'$  has all three possibilities.

(a) Case  $cab, cab$ :

$$\mathbf{w} \sim \mathbf{x}_1 \mathbf{y}_1 c \mathbf{x}_2 a \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 \sim \mathbf{y}_1 \mathbf{x}_1 c \mathbf{x}_2 a \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 = \\ \mathbf{y}_1 p_1(\mathbf{w}) \mathbf{y}_2 b \mathbf{y}_3 \sim \mathbf{y}_1 p_1(\mathbf{w}') \mathbf{y}_2 b \mathbf{y}_3 = \mathbf{y}_1 \tilde{\mathbf{x}}_1 c \tilde{\mathbf{x}}_2 a \tilde{\mathbf{x}}_3 \mathbf{y}_2 b \mathbf{y}_3.$$

Now we need the assumption that  $(b, c, \Gamma_2)$  is stable for a series of words taking  $\mathbf{y}_1 c \mathbf{y}_2 b \mathbf{y}_3$  to  $\tilde{\mathbf{y}}_1 c \tilde{\mathbf{y}}_2 b \tilde{\mathbf{y}}_3$  without using the relation  $ac = ca$ . By using relations of the type  $xy = yx$  we can use the same relations to obtain a sequence of words taking  $\mathbf{y}_1 \tilde{\mathbf{x}}_1 c \tilde{\mathbf{x}}_2 a \tilde{\mathbf{x}}_3 \mathbf{y}_2 b \mathbf{y}_3$  to  $\tilde{\mathbf{y}}_1 \tilde{\mathbf{x}}_1 c \tilde{\mathbf{x}}_2 a \tilde{\mathbf{x}}_3 \tilde{\mathbf{y}}_2 b \tilde{\mathbf{y}}_3 \sim \mathbf{w}'$  without using the relation  $ab = ba$ .

(b) Case  $cab, acb$ :

$$\mathbf{w} \sim \mathbf{x}_1 \mathbf{y}_1 c \mathbf{x}_2 a \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 \sim \mathbf{y}_1 \mathbf{x}_1 c \mathbf{x}_2 a \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 = \mathbf{y}_1 p_1(\mathbf{w}) \mathbf{y}_2 b \mathbf{y}_3 \sim \\ \mathbf{y}_1 p_1(\mathbf{w}') \mathbf{y}_2 b \mathbf{y}_3 = \mathbf{y}_1 \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 c \tilde{\mathbf{x}}_3 \mathbf{y}_2 b \mathbf{y}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 \mathbf{y}_1 c \mathbf{y}_2 b \mathbf{y}_3 \tilde{\mathbf{x}}_3 = \\ \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 p_2(\mathbf{w}) \tilde{\mathbf{x}}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 p_2(\mathbf{w}') \tilde{\mathbf{x}}_3 = \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_1 c \tilde{\mathbf{y}}_2 b \tilde{\mathbf{y}}_3 \tilde{\mathbf{x}}_3 \sim \mathbf{w}'$$

(c) Case  $cab, abc$ :

$$\mathbf{w} \sim \mathbf{x}_1 \mathbf{y}_1 c \mathbf{x}_2 a \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 \sim \mathbf{y}_1 \mathbf{x}_1 c \mathbf{x}_2 a \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 = \mathbf{y}_1 p_1(\mathbf{w}) \mathbf{y}_2 b \mathbf{y}_3 \sim \\ \mathbf{y}_1 p_1(\mathbf{w}') \mathbf{y}_2 b \mathbf{y}_3 = \mathbf{y}_1 \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 c \tilde{\mathbf{x}}_3 \mathbf{y}_2 b \mathbf{y}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 \mathbf{y}_1 c \mathbf{y}_2 b \mathbf{y}_3 \tilde{\mathbf{x}}_3 = \\ \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 p_2(\mathbf{w}) \tilde{\mathbf{x}}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 p_2(\mathbf{w}') \tilde{\mathbf{x}}_3 = \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_1 c \tilde{\mathbf{y}}_2 b \tilde{\mathbf{y}}_3 \tilde{\mathbf{x}}_3 \sim \mathbf{w}'$$

(d) Case  $acb, cab$ : See case  $cab, acb$  backwards.

(e) Case  $acb, acb$ :

$$\mathbf{w} \sim \mathbf{x}_1 \mathbf{y}_1 a \mathbf{x}_2 c \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 \sim \mathbf{y}_1 \mathbf{x}_1 a \mathbf{x}_2 c \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 = \mathbf{y}_1 p_1(\mathbf{w}) b \mathbf{y}_3 \sim \\ \mathbf{y}_1 p_1(\mathbf{w}') b \mathbf{y}_3 = \mathbf{y}_1 \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 c \tilde{\mathbf{x}}_3 \mathbf{y}_2 b \mathbf{y}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 \mathbf{y}_1 c \mathbf{y}_2 b \mathbf{y}_3 \tilde{\mathbf{x}}_3 = \\ \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 p_2(\mathbf{w}) \tilde{\mathbf{x}}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 p_2(\mathbf{w}') \tilde{\mathbf{x}}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_1 c \tilde{\mathbf{y}}_2 b \tilde{\mathbf{y}}_3 \tilde{\mathbf{x}}_3 \sim \mathbf{w}'$$

(f) Case  $acb, abc$ :

$$\mathbf{w} \sim \mathbf{x}_1 \mathbf{y}_1 a \mathbf{x}_2 c \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 \sim \mathbf{y}_1 \mathbf{x}_1 a \mathbf{x}_2 c \mathbf{x}_3 \mathbf{y}_2 b \mathbf{y}_3 = \mathbf{y}_1 p_1(\mathbf{w}) b \mathbf{y}_3 \sim \\ \mathbf{y}_1 p_1(\mathbf{w}') b \mathbf{y}_3 = \mathbf{y}_1 \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 c \tilde{\mathbf{x}}_3 \mathbf{y}_2 b \mathbf{y}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 \mathbf{y}_1 c \mathbf{y}_2 b \mathbf{y}_3 \tilde{\mathbf{x}}_3 = \\ \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 p_2(\mathbf{w}) \tilde{\mathbf{x}}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 p_2(\mathbf{w}') \tilde{\mathbf{x}}_3 \sim \tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_1 c \tilde{\mathbf{y}}_2 b \tilde{\mathbf{y}}_3 \tilde{\mathbf{x}}_3 \sim \mathbf{w}'$$

- (5) We have that  $\mathbf{w} \sim \mathbf{x}_1 \mathbf{y}_2 a \mathbf{x}_2 \mathbf{y}_2 b \mathbf{x}_3 \mathbf{y}_3$ . By assumption (of stability) we have a sequence of words taking  $\mathbf{x}_1 a \mathbf{x}_2 b \mathbf{x}_3$  to  $\tilde{\mathbf{x}}_1 a \tilde{\mathbf{x}}_2 b \tilde{\mathbf{x}}_3$  without using the relation  $ab = ba$ . By using extra relations of the type  $xy = yx$  we can use the same relations to get a sequence of words from  $\mathbf{x}_1 \mathbf{y}_2 a \mathbf{x}_2 \mathbf{y}_2 b \mathbf{x}_3 \mathbf{y}_3$  to  $\tilde{\mathbf{x}}_1 \mathbf{y}_2 a \tilde{\mathbf{x}}_2 \mathbf{y}_2 b \tilde{\mathbf{x}}_3 \mathbf{y}_3$ . By symmetry we may do the same for  $\mathbf{y}_i$ , which gives us a sequence of words from  $\mathbf{w}$  to  $\mathbf{w}'$  not using the relation  $ab = ba$ .  $\square$

**Lemma 44.** *Let  $a \in \Gamma$  be a source and  $b \in \Gamma$  be a sink. Assume that  $a \rightarrow b \in E(\Gamma)$  and let  $\Gamma' = \Gamma \setminus \{a \rightarrow b\}$ . Moreover assume that  $(a, b\Gamma')$  is stable. Then for each element  $[\mathbf{w}] \in M(\Gamma)$  we can assign a pair  $([\widetilde{\mathbf{w}}], s) \in M(\Gamma') \times \{+, -\}$  such that  $s$  represents the order of  $a$  and  $b$  and  $[\widetilde{\mathbf{w}}] = [\mathbf{w}]/(aba = bab = ab)$ . This is injective.*

*Proof.* Given  $[\widetilde{\mathbf{w}}] \in M(\Gamma')$  there are three mutually exclusive possibilities:

- (1) all words in  $[\widetilde{\mathbf{w}}]$  which have  $a, b$  multiplicity free have  $a$  before  $b$

- (2) all words in  $[\widetilde{\mathbf{w}}]$  which have  $a, b$  multiplicity free have  $b$  before  $a$
- (3) both  $a$  before  $b$  and  $b$  before  $a$  appears in words  $\mathbf{w} \in [\widetilde{\mathbf{w}}]$

The first two cases are clear. Thus what remains to show is injectivity in the third case. Let's choose one word  $\mathbf{w} \in [\widetilde{\mathbf{w}}] \cap \mathcal{MF}_{\{a,b\}}$ , such that  $a, b$  come in the desired order, and consider  $[\mathbf{w}] \in M(\Gamma)$ . Let  $\tilde{\mathbf{w}} \in [\mathbf{w}]$ . Then there is a series of words  $\mathbf{w} = \mathbf{w}_1 \approx \dots \approx \mathbf{w}_k = \tilde{\mathbf{w}}$ . By lemma 38 we may assume that  $\mathbf{w}_i \in \mathcal{MF}_{\{a,b\}}$  for all  $i$ . By assumption we have can assume that  $a$  and  $b$  do not switch places. Thus can lift the series of equalities to be of words of elements in  $HK_\Gamma$ .  $\square$

**Proposition 45.** *Let  $\Gamma$  be as above (including that  $(a, b, \Gamma')$  is stable) and let  $R_{f'}$  be an effective representation. Then any extension  $f$  of  $f'$  which has  $f_{ab} := f(a \rightarrow b) \neq 0$  gives an effective representation  $R_f$  of  $HK_\Gamma$ .*

*Proof.* We need to construct the pair  $([\widetilde{\mathbf{w}}], s)$  from a given matrix  $R_{f'}[\mathbf{w}]$ . We have that  $R_f[\mathbf{w}](x) = R_{f'}[\mathbf{w}](x)$  for all vertices  $x \neq b$  and

- (1) If  $R_f[\mathbf{w}](b) = 0$  then  $R_{f'}[\mathbf{w}](b) = 0$  and  $s = -$
- (2) If  $R_f[\mathbf{w}](b) = \sum c_x x$  then  $R_{f'}[\mathbf{w}](b) = \sum c_x x$  and  $s = -$
- (3) If  $R_f[\mathbf{w}](b) = f_{ab}a + \sum c_x x$  then  $R_{f'}[\mathbf{w}](b) = \sum c_x x$  and  $s = + \forall x \in V(\Gamma)$

Note that this covers all possibilities. Since  $R_{f'}$  is assumed to be effective, we get that  $R_f$  is effective from the previous lemma.  $\square$

**Proposition 46.** *Let  $\Gamma = \Gamma_1 \cup \Gamma_2$  be such that  $\Gamma_1 \cap \Gamma_2 = \{a, b\}$  where  $a, b$  are kinks in  $\Gamma$  and  $(a, b, \Gamma_1), (a, b, \Gamma_2)$  are stable. Then the projection  $p: HK_\Gamma \rightarrow HK_{\Gamma_1} \times HK_{\Gamma_2}$  is injective.*

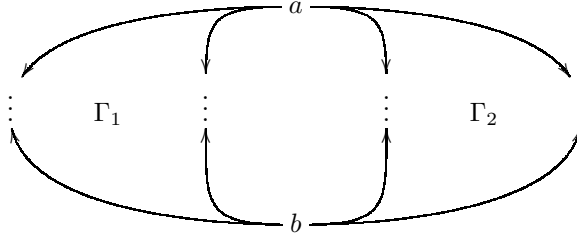


FIGURE 4.1.  $\Gamma = \Gamma_1 \cup \Gamma_2$

*Proof.* We can always choose a pair of words  $(\mathbf{w}_1, \mathbf{w}_2) \in p([\mathbf{w}])$  such that  $a, b$  are multiplicity free in  $\mathbf{w}$  and the order is the same in both pairs, i.e.  $\mathbf{w}_1 = \mathbf{x}_1\pi(a)\mathbf{x}_2(b)\mathbf{x}_3, \mathbf{w}_2 = \mathbf{y}_1\pi(a)\mathbf{y}_2\pi(b)\mathbf{y}_3$ . We then make the construction  $\mathbf{w} = \mathbf{x}_1\mathbf{y}_1\pi(a)\mathbf{x}_2\mathbf{y}_2\pi(b)\mathbf{x}_3\mathbf{y}_3$ . This construction is in fact independent of choice of words. To see this we need to separate in two cases, depending on if  $(\mathbf{w}_1, \mathbf{w}_2)$  and  $(\tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2)$  have the same permutation or not. W.l.o.g  $\pi \equiv \text{id}$ .

- (1) Same permutation. Thus we have  $\mathbf{x}_1a\mathbf{x}_2b\mathbf{x}_3 \sim \tilde{\mathbf{x}}_1a\tilde{\mathbf{x}}_2b\tilde{\mathbf{x}}_3$  and  $\mathbf{y}_1a\mathbf{y}_2b\mathbf{y}_3 \sim \tilde{\mathbf{y}}_1a\tilde{\mathbf{y}}_2b\tilde{\mathbf{y}}_3$ . We can use the fifth case of 43 to see that  $a, b, \Gamma$  is stable and thus obtain a sequence of words taking  $\mathbf{x}_1a\mathbf{x}_2b\mathbf{x}_3 \sim \tilde{\mathbf{x}}_1a\tilde{\mathbf{x}}_2b\tilde{\mathbf{x}}_3$  without using  $ab = ba$ . The same sequence can be used to get

$$\mathbf{w} = \mathbf{x}_1\mathbf{y}_1a\mathbf{x}_2\mathbf{y}_2b\mathbf{x}_3\mathbf{y}_3 \sim \dots \sim \tilde{\mathbf{x}}_1\mathbf{y}_1a\tilde{\mathbf{x}}_2\mathbf{y}_2b\tilde{\mathbf{x}}_3\mathbf{y}_3$$

and, in a similar fashion,

$$\mathbf{w} = \mathbf{x}_1 \mathbf{y}_1 a \mathbf{x}_2 \mathbf{y}_2 b \mathbf{x}_3 \mathbf{y}_3 \sim \dots \sim \tilde{\mathbf{x}}_1 \mathbf{y}_1 a \tilde{\mathbf{x}}_2 \mathbf{y}_2 b \tilde{\mathbf{x}}_3 \mathbf{y}_3 \sim \dots \sim \tilde{\mathbf{x}}_1 \tilde{\mathbf{y}}_1 a \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_2 b \tilde{\mathbf{x}}_3 \tilde{\mathbf{y}}_3 = \tilde{\mathbf{w}}.$$

- (2) Different permutation. If  $\mathbf{x}_1 a \mathbf{x}_2 b \mathbf{x}_3 \sim \tilde{\mathbf{x}}_1 b \tilde{\mathbf{x}}_2 a \tilde{\mathbf{x}}_3$  we have that  $a$  and  $b$  only have to switch order once. At the point where  $a$  and  $b$  meet we have  $\mathbf{x}_1 a b \mathbf{x}_2 = \mathbf{x}_1 b a \mathbf{x}_2$  and  $\mathbf{y}_1 a b \mathbf{y}_2 = \mathbf{y}_1 b a \mathbf{y}_2$ . By the case above, we may assume that  $\mathbf{w} = \mathbf{x}_1 \mathbf{y}_1 a b \mathbf{x}_2 \mathbf{y}_2$  and  $\tilde{\mathbf{w}} = \mathbf{x}_1 \mathbf{y}_1 b a \mathbf{x}_2 \mathbf{y}_2$ . This gives

$$\mathbf{w} = \mathbf{x}_1 \mathbf{y}_1 a b \mathbf{x}_2 \mathbf{y}_2 \sim \mathbf{y}_1 \mathbf{x}_1 a b \mathbf{x}_2 \mathbf{y}_2 \approx \mathbf{y}_1 \mathbf{x}_1 b a \mathbf{x}_2 \mathbf{y}_2 \sim \mathbf{x}_1 \mathbf{y}_1 b a \mathbf{x}_2 \mathbf{y}_2 = \tilde{\mathbf{w}}.$$

□

**Corollary 47.** *Assume conjecture 42 is true. Given  $\Gamma$  as above, assume that  $R_{f_1}, R_{f_2}$  are effective and that there is no directed path from  $a$  to  $b$  or vice versa. Then the representation  $R_f$ , up induced from  $\{R_{f_1}, R_{f_2}\}$ , is effective.*

*Proof.* Given  $R_f[\mathbf{w}]$  we use theorem 18 to get  $R_{f_1} p_1[\mathbf{w}]$  and  $R_{f_2} p_2[\mathbf{w}]$ . Since  $R_{f_1}$  and  $R_{f_2}$  were assumed to be effective we get a unique pair  $(p_1[\mathbf{w}], p_2[\mathbf{w}])$ , which we feed into the previous theorem to obtain a  $[\mathbf{w}']$  uniquely defined from  $R_f[\mathbf{w}]$ . □

**Definition 48.** A *closed chain* is a graph  $\Gamma$  which has a cyclic connections of links, i.e. it can be obtained from a chain by joining the outermost links in a kink. We will enumerate the links and the kinks with  $\mathbb{Z}_n$  such that  $a_i \in \Gamma_i \cap \Gamma_{i+1}$ . We require the number of links to be at least three.

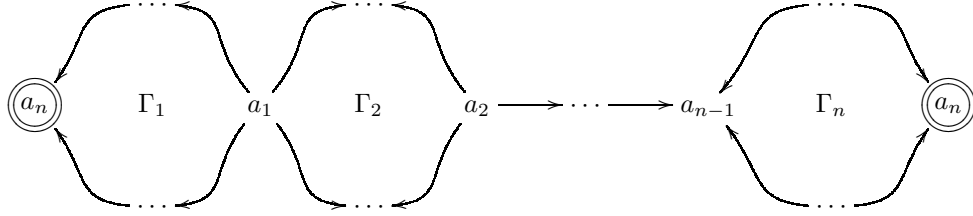


FIGURE 4.2. How a closed chain might look. Note that the leftmost and the rightmost vertices are the same.

We may define the projection  $p$  onto links for closed chains as well. To prove injectivity we make use of blocks as before, but we need a slightly more complicated construction, as follows

- (1) Choose a word  $\mathbf{w}_i \in (V(\Gamma_i))^*$  which is multiplicity free with respect to  $a_{i-1}$  and  $a_i$  for each link  $\Gamma_i$ . In general we have to make the choices dependantly because we need to make sure that there is a (global) permutation which agrees with the orders of the kinks in each word. However, since we are only interested in  $\text{Im}(p)$  there is at least one dependant choice we can make.
- (2) For  $\mathbf{w}_i = \mathbf{x}_1 \alpha_{\pi_i(i-1)} \mathbf{x}_2 \alpha_{\pi_i(i)} \mathbf{x}_3$  and  $\mathbf{w}_{i+1} = \mathbf{y}_1 \alpha_{\pi_{i+1}(i)} \mathbf{y}_2 \alpha_{\pi_{i+1}(i+1)} \mathbf{y}_3$  we construct the block  $A_i = \tilde{\mathbf{x}}_1 \tilde{\mathbf{y}}_1 \alpha_i \tilde{\mathbf{x}}_2 \tilde{\mathbf{y}}_2$  where

$$(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2) = \begin{cases} (\varepsilon, \mathbf{x}_3), & \pi_i = \text{id} \\ (\mathbf{x}_1, \mathbf{x}_2), & \pi_i \neq \text{id} \end{cases}$$

$$(\bar{\mathbf{y}}_1, \bar{\mathbf{y}}_2) = \begin{cases} (\mathbf{y}_1, \mathbf{y}_2), & \pi_{i+1} = \text{id} \\ (\varepsilon, \mathbf{y}_3), & \pi_{i+1} \neq \text{id} \end{cases}$$

- (3) We choose one permutation  $\pi$  such that  $\pi(i) < \pi(i+1) \iff \pi_{i+1}(i) < \pi_{i+1}(i+1)$  for all  $i$ .  
 (4) Set  $\mathbf{w} = A_{\pi(1)} A_{\pi(2)} \cdots A_{\pi(n)}$ .

**Proposition 49.** *Let  $\Gamma$  be a closed chain with at least three links such that every link is stable with the connecting kinks. Then  $p$  is injective and  $p^{-1} : \text{Im}(p) \rightarrow HK_\Gamma$  is given by the construction above.*

*Proof.* It is quite clear that the construction above defines an inverse to  $p$  if it is well-defined as a map from  $\text{Im}(p)$  to  $HK_\Gamma$ . Indeed if  $\pi_i = \text{id}$  we have

$$\mathbf{w} = A_{\pi(1)} \cdots A_{i-1} \cdots A_i \cdots A_{\pi(n)} = A_{\pi(1)} \cdots \bar{\mathbf{x}}_1 \mathbf{y}_1 \alpha_i \bar{\mathbf{x}}_2 \mathbf{y}_2 \cdots \bar{\mathbf{z}}_1 \alpha_i \mathbf{y}_3 \bar{\mathbf{z}}_2 \cdots A_{\pi(n)},$$

which gives  $p[\mathbf{w}] = (\cdots, [\mathbf{y}_1 \alpha_i \mathbf{y}_2 \alpha_2 \mathbf{y}_3], \cdots) = (\cdots, [\mathbf{w}_i], \cdots)$ . The case  $\pi \neq \text{id}$  is proved similarly.

It is enough to prove the claim for a chain with three links, because we may join extra links together with theorem 20. The projections agree on such subgraphs, so we may view a chain as a link in a closed chain. So we need to prove that the construction gives a well-defined map  $p^{-1}$ .

We may translate and possibly reflect the permutation  $\pi$  to assume  $\pi = \text{id}$ , i.e.  $\mathbf{w} = A_1 A_2 A_3$ . Now we need to study different cases.  $\square$

- (1) Case  $\tilde{\pi}(1) = 1$ . We use the relation  $\mathbf{w}_3 \sim \tilde{\mathbf{w}}_3$  and the technique in lemma 32 to get  $\tilde{\mathbf{w}} = \tilde{A}_1 \tilde{A}_{\tilde{\pi}(2)} \tilde{A}_{\tilde{\pi}(3)} \sim \tilde{A}_1 \tilde{A}_2 \tilde{A}_3$ . Thus we may assume  $\tilde{\pi} = \text{id}$ . Again we use the technique from lemma 32 to get

$$\begin{aligned} A_1 A_2 A_3 &= (\mathbf{z}_1 \mathbf{x}_1 \alpha_1 \mathbf{z}_2 \mathbf{x}_2) (\mathbf{x}_3 \mathbf{y}_1 \alpha_2 \mathbf{x}_4 \mathbf{y}_2) (\mathbf{y}_3 \mathbf{z}_3 \alpha_3 \mathbf{y}_4 \mathbf{z}_4) \sim \cdots \\ &\sim (\mathbf{z}_1 \tilde{\mathbf{x}}_1 \alpha_1 \mathbf{z}_2 \tilde{\mathbf{x}}_2) (\tilde{\mathbf{x}}_3 \mathbf{y}_1 \alpha_2 \tilde{\mathbf{x}}_4 \mathbf{y}_2) (\mathbf{y}_3 \mathbf{z}_3 \alpha_3 \mathbf{y}_4 \mathbf{z}_4) \sim \cdots \\ &\sim (\mathbf{z}_1 \tilde{\mathbf{x}}_1 \alpha_1 \mathbf{z}_2 \tilde{\mathbf{x}}_2) (\tilde{\mathbf{x}}_3 \tilde{\mathbf{y}}_1 \alpha_2 \tilde{\mathbf{x}}_4 \tilde{\mathbf{y}}_2) (\tilde{\mathbf{y}}_3 \mathbf{z}_3 \alpha_3 \tilde{\mathbf{y}}_4 \mathbf{z}_4) \end{aligned}$$

Now all we need to do is to change  $\mathbf{z}_i$  to  $\tilde{\mathbf{z}}_i$ . We may find a series of words  $\mathbf{w}_3 = \mathbf{w}_{3,1} \approx \mathbf{w}_{3,2} \approx \cdots \approx \mathbf{w}_{3,k} = \tilde{\mathbf{w}}_3$  such that  $\mathbf{w}_{3,i}$  is multiplicity free with respect to  $a_1, a_3$  and such that the relation  $a_1 a_3 = a_3 a_1$  is not used, by assumption. Since the relations in the sequence  $\mathbf{w}_{3,i}$  uses at most one of  $a_1, a_3$  and all vertices in  $V(\Gamma_3 \setminus \{a_1, a_3\})$  commute with all vertices in  $V(\Gamma \setminus \Gamma_3)$  we may commute the vertices in  $\mathbf{z}_i$  to  $a_1$  or  $a_3$  if that is where the relation was used.

- (2) Case  $\tilde{\pi}(1) = 2$ . Let  $\tilde{\pi}(i) = 1$ . Depending on  $i = 1$  or  $3$  we have  $a_1, a_i$  in  $\mathbf{w}_1$  or in  $\mathbf{w}_3$ . Let  $\mathbf{w}_j$  be the appropriate word. We have  $\mathbf{w}_j = \mathbf{u}_1 a_1 \mathbf{u}_2 a_i \mathbf{u}_3$  and  $\tilde{\mathbf{w}}_j = \tilde{\mathbf{u}}_1 a_i \tilde{\mathbf{u}}_2 a_1 \tilde{\mathbf{u}}_3$ . By using this relation and the technique from lemma 32, case different permutation we may get  $\tilde{A}_i \tilde{A}_1 \sim \tilde{A}_1 \tilde{A}_i$ , which reduces to the case  $\hat{\pi}(1) = 1$ .  
 (3) Case  $\tilde{\pi}(1) = 3$ . Use a similar argument as above to get to  $\hat{\pi}(1) = 2$ .

**Corollary 50.** *Let  $\Gamma$  be a closed chain with at least three links such that  $(a_{i-1}, a_i, \Gamma_i)$  is stable for all  $i$ . Assume that every  $R_{f_i}$  is an effective representation of  $HK_{\Gamma_i}$ , respectively, and that  $R_f$  is up induced from  $\{R_{f_i}\}$ . Then  $R_f$  is an effective representation of  $HK_\Gamma$ .*

*Proof.* We begin by joining extra links together so that we end up with a closed chain with three links. By the fourth case of Proposition 43 we still will have that all  $(a_{i-1}, a_i, \Gamma_i)$  are stable for the “new” closed chain. We may refer to the chain

case to see that all  $R_{f_i}$  also are effective. Thus we may assume that  $\Gamma$  has exactly three links.

From  $R_f[\mathbf{w}]$  we get  $R_{f_1}[\mathbf{w}]$ ,  $R_{f_2}[\mathbf{w}]$  and  $R_{f_3}[\mathbf{w}]$  (uniquely). Since  $R_{f_i}$  is assumed to be effective, we get  $[\mathbf{w}_1]$ ,  $[\mathbf{w}_2]$  and  $[\mathbf{w}_3]$  uniquely. But  $p$  is injective, so there exists a unique  $[\mathbf{w}'] = p^{-1}([\mathbf{w}_1], [\mathbf{w}_2], [\mathbf{w}_3])$  and we have  $[\mathbf{w}] = [\mathbf{w}']$  because  $R_f[\mathbf{w}']$  is the same endomorphism as  $R_f[\mathbf{w}]$ .  $\square$

#### REFERENCES

- [1] S. Alsaody, *Determining the elements of a semigroup*, U.U.D.M. Report 2007:3, ISSN 1101-3591, Dept. of Mathematics, Uppsala University.
- [2] A. Gensing, *Monoid algebras of projection functors*, preprint *arXiv*: <http://arxiv.org/pdf/1203.1943v1.pdf> (2012)
- [3] Ch. O. Kiselman, *A semigroup of operators in convexity theory*, Trans. Amer.Math. Soc. 354 (2002), no. 5, 2035-2053.
- [4] O. Ganyushkin and V. Mazorchuk, *On Kiselman quotients of 0-Hecke monoids*, International Electronic J. of Alg. (2011), 174-191.
- [5] G. Kudryavtseva and V. Mazorchuk, *On Kiselman's semigroup*, Yokohama Math. J., 55(1) (2009), 21-46.
- [6] V. Mazorchuk and B. Steinberg, *Effective dimension of finite semigroups*, preprint *arXiv*: <http://arxiv.org/pdf/1107.1413v3.pdf> (2011), to appear in J. Pure Appl. Alg.
- [7] A. Paasch, *Monoidalgebren von Projektionsfunktoren*, Ph.D. thesis, Bergischen Universität Wuppertal, <http://elpub.bib.uni-wuppertal.de/servlets/DerivateServlet/Derivate-2480/dc1113.pdf> (2011), german
- [8] The on-line encyclopedia of integer sequences, <https://oeis.org/>